

# The chart based approach to studying the global structure of a spacetime induces a coordinate invariant boundary.

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I demonstrate that the chart based approach to the study of the global structure of Lorentzian manifolds induces a homeomorphism of the manifold into a topological space as an open dense set. The topological boundary of this homeomorphism is a chart independent boundary of ideal points equipped with a topological structure and a physically motivated classification. I show that this new boundary contains all other boundaries that can be presented as the topological boundary of an envelopment. Hence, in particular, it is a generalisation of Penrose's conformal boundary. I provide three detailed examples: the conformal compactification of Minkowski spacetime, Scott and Szekeres' analysis of the Curzon singularity and Beyer and Hennig's analysis of smooth Gowdy symmetric generalised Taub-NUT spacetimes.

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## 1. Introduction

When presented with a spacetime and asked what its global structure is the usual approach is to construct charts that ‘extend to the edge of the manifold’ and then analyse the behaviour of coordinate invariant quantities in the charts, e.g. the Kretschmann scalar or a congruence of geodesics. The charts are then altered so that the invariant quantities have suitably nice behaviour in the limit to ‘the edge of the manifold’. The construction of the chart  $\beta$ , described in Section 3.2, is an example of this. The structure of these altered charts ‘on the edge’ is then taken to be a boundary for the spacetime, with justification coming from the nice behaviour of the particular invariants used. Minkowski spacetime, equipped with Penrose’s conformal boundary, viewed as a collection of coordinate transformations, is a well known example, see Section 3.1. The original justification was based on the conformal invariance of the zero rest-mass free-field equation for any spin value, [20, 22, 21].

Unfortunately this approach is not necessarily chart independent. Could other charts that also allow for nice behaviour of the invariants ‘on the edge’ exist? Could these charts give boundaries that have different topologies? Given a chart that ‘extends to the edge’ of a manifold and an invariant which has nice behaviour ‘on the edge’ do there exist other invariants which do not have nice behaviour ‘on the edge’? One of the motivators in the field of boundary constructions is the desire for a boundary that generalizes Penrose’s conformal boundary and avoids these coordinate dependence problems, [16]. See [13, Section 6] for further discussion of this and a review of boundary constructions in general.

Penrose’s conformal boundary, while well adapted to describing causal structure, fails to describe the gravitational field of an asymptotically flat space time which has non-zero ADM mass, [21, page 569]. The gravitational field is singular at both spacelike and past / future timelike infinities, [21, pages 568–569]. In the context of the discussion above: the conformal boundary is given by charts for which the invariant causal structure has nice behaviour ‘on the edge’. The same charts, however, fail to provide nice behaviour for the metric ‘on the edge’, when the spacetime has non-zero ADM mass. This is one of the reasons underlying Friedrich’s modifications of Penrose’s conformal boundary, [10, 11].

The definition of a coordinate independent boundary, that is easy to construct, fits with physical intuition and generalises some appropriate properties of Penrose’s confor-

mal boundary, has proven to be very difficult, [1, Section 2], [27, Section 2.3] and [13]. The literature contains numerous suggested boundaries, examples of how they behave badly and their subsequent alterations, [1, Section 2] and [27, Section 2.3]. As a result the boundaries that are still being developed are sophisticated and technical, [13, 25, 9, 12].

The apparently necessary technicalities of boundary constructions divorces them from the more natural chart based approach. As a consequence of this the field of boundary constructions has become separated from the main stream of research in Relativity. Penrose’s conformal boundary is a notable exception. It has been successfully applied in many areas, [13]. This paper is motivated by two of the reasons why; (i) it can be constructed using charts and (ii) it can be presented as the topological boundary of an envelopment of the manifold. This gives the boundary a convenient construction and a differential structure. In particular, calculations on the conformal boundary can be performed using charts in some larger manifold. Coordinate independence is then ensured by requiring that quantities defined on the boundary transform in the appropriate way with respect to the charts of the larger manifold.

In this paper I attempt to reconcile this divorce by showing that the chart based approach to the study of global structure equips the manifold with a chart independent boundary which; (i) contains (and therefore is a generalisation of) Penrose’s conformal boundary, see Section 2.4 and in particular Proposition 2.29, (ii) has a coordinate description, see Section 2.3 and the discussion following Definition 2.15, and (iii) allows for the use of differential structure on the boundary, see for example Proposition 2.8 and the discussion following Definition 2.15.

The boundary is based on the following idea. Each chart  $\alpha : U \subset M \rightarrow \mathbb{R}^n$ , with a suitable assumption on its behaviour, Definition 2.4, can be associated with some larger subset,  $V$ , of  $\mathbb{R}^n$  so that  $\alpha(U) \subset V$ . The topological boundary of  $\alpha(U)$  relative to  $V$ ,  $\partial_V \alpha(U)$ , can then be considered as a local representation of “the boundary of  $M$ ”. Because of the assumption on  $\alpha$ , the set  $V$  can be interpreted as the domain of some larger chart in some enveloping manifold of  $M$ , see Proposition 2.8. This equips the local representation of the boundary of  $M$ ,  $\partial_V \alpha(U)$ , with a differential structure. These details are described in Section 2.1 and the discussion after Definition 2.15.

Calculations performed in these charts can now be said to describe coordinate dependent quantities on local representations of the boundary of  $M$ . If the quantity transforms the right way under “coordinate transformations on the boundary” then it is possible to claim that the coordinate dependent quantities describe, with respect to the chosen chart, a coordinate independent object on the boundary. This idea is explored in Sections 2.2 and 2.3 where a topological space is constructed, Definition 2.15 (which gives a concrete realisation of the boundary of  $M$ ), and a precise definition of the coordinate transformations on the boundary is given, Proposition 2.17.

The topological boundary,  $\mathcal{B}_{\text{ch}}(M)$ , of  $M$  in this topological space is the chart independent object that the local representations,  $\partial_V \alpha(U)$ , represent. Given some quantity that is well defined on some local representation of  $\mathcal{B}_{\text{ch}}(M)$ , it is very likely that there will exist some other local representation in which that quantity is not well defined. This can be thought of as a manifestation of the same issue that the conformal boundary has with gravitational fields of spacetimes with non-zero ADM mass. However, the freedom

to choose local representations of the boundary that are appropriate for the quantity being studied allow for  $\mathcal{B}_{\text{ch}}(M)$  to cope with this issue.

This is such an important point that a digression is warranted. The construction of  $\mathcal{B}_{\text{ch}}(M)$  makes only one topological restriction (Definition 2.4) on the charts used. In particular, no geometric or physical information is used. This ensures that  $\mathcal{B}_{\text{ch}}(M)$  is uniquely associated to any manifold and frees it from claims of arbitrariness. The downside is that, in general, explicit construction of  $\mathcal{B}_{\text{ch}}(M)$  can never be completed. This is, however, not needed as calculations on local representations of  $\mathcal{B}_{\text{ch}}(M)$ , once checked for coordinate independence on  $\mathcal{B}_{\text{ch}}(M)$ , are sufficient to produce well defined quantities on  $\mathcal{B}_{\text{ch}}(M)$ . In this way the standard chart based approach to the study of global structure can be freed from claims of coordinate dependence. Because the construction of  $\mathcal{B}_{\text{ch}}(M)$  makes only a mild topological restriction on charts this boundary provides no guidance on which charts are suitable for the calculation of particular quantities. This is a point that I shall discuss in more detail below and in Section 3.

Section 2.2 defines an equivalence relation, Definition 2.13, which describes how the different local representations of the boundary of  $M$  are related to each other. Given two charts,  $\alpha : U \subset M \rightarrow \mathbb{R}^n$  and  $\beta : V \subset M \rightarrow \mathbb{R}^n$  and  $x \in U \cap V$  then  $x$  has the two representations  $p = \alpha(x) \in \mathbb{R}^n$  and  $q = \beta(x) \in \mathbb{R}^n$ . We know that  $p$  and  $q$  are two different representations of some third, chart independent, point  $x$ . If we did not have the object  $x$ , but only the functions  $\beta$  and  $\alpha$  as well as the points  $p$  and  $q$ , then it is still possible to determine if  $p$  and  $q$  represent some third, chart independent, point by asking if  $\beta \circ \alpha^{-1}(p) = q$ . This defines an equivalence relation on the collection of all charts and points in the images of the charts. The extension of this equivalence relation to local representations of the boundary of  $M$  is exactly the equivalence relation given in Definition 2.13.

With the equivalence relation it is possible to construct a topological space into which the manifold is embedded as an open dense set, see Section 2.3. Continuing the analogy described above, if  $p \in \alpha(U)$  and  $q \in \beta(V)$  are equivalent then the third, chart independent, point  $x$  can be described as the equivalence class,  $[(\alpha, p)]$ , of  $(\alpha, p)$  such that  $(\beta, q) \in [(\alpha, p)]$  if and only if  $\beta \circ \alpha^{-1}(p) = q$ . The topological space is composed of these equivalence classes for the extension of the equivalence relation to the local representations of the boundary. It is in this sense that the topological space is coordinate independent. The definition of the topological space, Definitions 2.15 and 2.19, is as a quotient space of the disjoint union of  $\alpha(U) \cup \partial_V \alpha(U)$  over all allowable charts  $\alpha$ , see Definition 2.4. The quotient map has several nice properties, see the discussion after Definition 2.15 and Appendix A. In particular the quotient map, restricted to  $\alpha(U) \cup \partial_V \alpha(U)$  for any allowable chart, is a homeomorphism. This implies that the boundary,  $\mathcal{B}_{\text{ch}}(M)$ , of  $M$  has a local description in terms of charts, mediated by these homeomorphisms, so that the induced local representations (described in Section 2.1) also carry differential structure (via Proposition 2.8). In this way motivations (ii) and (iii) are justified.

With  $\mathcal{B}_{\text{ch}}(M)$  defined it is possible to define the “coordinate transformations on the boundary”, see Proposition 2.17. These coordinate transformations are homeomorphisms on the disjoint union used to define the topological space. They are extensions of

the original coordinate transformations on  $M$  to the local representations of the boundary of  $M$ . While the topological space itself is, in general (see Corollary 2.30), not a manifold the coordinate transformations on the boundary do have a close relationship to genuine coordinate transformations in some larger manifold. This relationship is described in Section 2.4.

Propositions 2.18 and 2.29 and Corollary 2.30 show that if the manifold  $M$  can be embedded into some larger manifold  $M_\phi$  by a map  $\phi : M \rightarrow M_\phi$  then the topological boundary of the image of  $\phi$ ,  $\partial\phi(M)$ , can be considered to be a subset of the boundary,  $\mathcal{B}_{\text{ch}}(M)$ , of  $M$ . Since Penrose’s conformal boundary can be presented in such a way these results justify motivation (i), see the comments after Corollary 2.30.

The boundary of  $M$ ,  $\mathcal{B}_{\text{ch}}(M)$ , is similar to the Abstract Boundary, see Proposition 2.23 and the comments after Proposition 2.23. As a consequence it is possible to apply Scott and Szekeres’ classification of Abstract Boundary points to  $\mathcal{B}_{\text{ch}}(M)$ . This is done in Section 2.5. Note that while  $\mathcal{B}_{\text{ch}}(M)$  and the Abstract Boundary are similar,  $\mathcal{B}_{\text{ch}}(M)$  is richer due to the presence of the topological space and the expression of the equivalence relation in terms of homeomorphisms that extend certain coordinate transformations on  $M$ . Because the topological space is defined via a quotient map the topology on  $\mathcal{B}_{\text{ch}}(M)$  can be studied using the vast number of existing results on quotient spaces (in contrast with the topology given in [2]). On a more technical note, each element of the boundary defined here is an equivalence class. These equivalence classes, due to the use of the quotient space construction, contain only boundary points not boundary sets, see Definition 2.7. Because of this certain technicalities regarding the Abstract Boundary, e.g. [27, Chapter 9], can be avoided.

Section 3 presents three examples of the construction and classification of representations of the boundary; the conformal compactification of Minkowski spacetime as it is given in [18, Section 5.1] (note that this presentation of the boundary differs from Penrose’s own publications [20, 22, 21], in particular it includes spacelike and timelike infinity); Scott and Szekeres’ construction of the maximal extension of the Curzon solution [23, 24] and Beyer and Hennig’s analysis of the global structure of smooth Gowdy symmetric generalized Taub-NUT solutions, [5]. The examples were selected to demonstrate application of the material in Section 2 and illustrate how standard global analysis using charts translates into information about the boundary. It is worthwhile noting that Beyer and Hennig’s analysis does not involve a closed form of the metric and as a consequence additional work would be required before other boundary constructions could be applied to their class of spacetimes.

The charts used in Section 3 were constructed to provide explanatory (and, perhaps, predictive) power regarding the global structure of the particular manifolds considered. To do this the behaviour of geometric and physical quantities on or near local representations of  $\mathcal{B}_{\text{ch}}(M)$  were studied, [20, 22, 21, 23, 24, 5]. As mentioned above, the construction of  $\mathcal{B}_{\text{ch}}(M)$  provides no guidance on the construction of such charts, rather the existing techniques of the chart based study of global structure should be used. The boundary  $\mathcal{B}_{\text{ch}}(M)$  then provides a way of comparing information computed in different charts on “the boundary of the manifold”, hence providing a method via which claims of coordinate dependence can be counted.

The paper closes with two Appendices. Appendix [A](#) contains proofs of claims made in Section [2.3](#) and Appendix [B](#) gives the proofs of two results stated in Section [2.4](#). The results and claims that rest on the material in these appendices are clearly highlighted.

### 1.1. Notation and Definitions

The disjoint union of two sets  $U, V$  is denoted by  $U \sqcup V$ .

Given a topological space,  $X$ , and  $U \subset X$  the boundary  $\overline{U} \setminus \text{int } U$  will be denoted by  $\partial U$ . This is the topological boundary of  $U$  in  $X$ . Given  $U \subset V \subset X$  the topological boundary of  $U$  relative to  $V$  will be denoted by  $\partial_V U$ . When  $U$  and  $V$  are open in  $X$ ,  $\partial_V U = V \cap \partial U$ . Sequences in  $X$  are denoted by  $(x_i) \subset X$ , where it is implicitly assumed that  $i \in \mathbb{N}$ . If the sequence  $(x_i)$  converges to  $x$ , I shall write  $x_i \rightarrow x$ .

A topological manifold is a locally Euclidean, second-countable, Hausdorff topological space. Note that some authors, e.g. [\[14\]](#), drop second-countability. A manifold,  $M$ , is a second countable, Hausdorff, topological space equipped with a maximal  $C^\infty$  atlas  $\mathcal{A}(M)$  of functions from  $M$  to  $\mathbb{R}^n$  where  $n$  denotes the dimension of  $M$ . In order to reduce the proliferation of notation the domain of a chart  $\alpha \in \mathcal{A}(M)$  will be denoted by  $\text{dom}(\alpha)$  and the range of  $\alpha$  by  $\text{ran}(\alpha)$ . I will always assume that  $\text{dom}(\alpha) \subset M$ ,  $\text{ran}(\alpha) \subset \mathbb{R}^n$  and that both sets are open. The phrase ‘manifold with boundary’ is used exclusively to denote an  $n$ -dimensional manifold with  $C^\infty$  atlas from  $M$  to  $\mathbb{R}^{n-1} \times \{x \in \mathbb{R} : x \geq 0\}$ . A topological manifold with boundary is a second-countable, Hausdorff topological space that is locally homeomorphic to  $\mathbb{R}^{n-1} \times \{x \in \mathbb{R} : x \geq 0\}$ .

An embedding,  $\phi : M \rightarrow M_\phi$ , of  $M$  into a manifold  $M_\phi$  of the same dimension as  $M$  is called an envelopment of  $M$ . When  $\hat{M}$  and  $M$  are both manifolds,  $\hat{M} \subset M$  will always imply that  $\hat{M}$  is a regular submanifold of  $M$ . Given an embedding  $\phi : M \rightarrow M_\phi$  the topological boundary of  $\phi(M)$  in  $M_\phi$  is  $\partial\phi(M)$ , in accordance with the notation above.

A  $C^l$  pseudo-Riemannian manifold,  $(M, g)$ , is a manifold  $M$  equipped with a symmetric, non-degenerate,  $C^l$  two-form  $g$ . Given a  $C^l$  pseudo-Riemannian manifold  $(M, g)$  and a  $C^k$ ,  $k \leq l$ , pseudo-Riemannian manifold  $(\hat{M}, \hat{g})$ , the pair  $(\hat{M}, \hat{g})$  is a  $C^k$  extension of  $(M, g)$  if  $M \subset \hat{M}$  and  $\hat{g}|_{TM \times TM} = g$ .

A curve,  $\gamma : I \rightarrow M$ , is a  $C^0$ , piecewise  $C^1$ , function from an interval  $I \subset \mathbb{R}$  to  $M$ . When expressing relationships between subsets or points of  $M$  and the image of  $\gamma$  the symbol  $\gamma$  rather than, the more correct,  $\gamma(I)$  will be used. Hence,  $p \in \gamma$  stands for  $p \in \gamma(I)$ ,  $p \in \overline{\gamma}$  stands for  $p \in \overline{\gamma(I)}$  and  $\gamma \cap \text{dom}(\alpha)$  stands for  $\gamma(I) \cap \text{dom}(\alpha)$ .

The curve  $\gamma$  is bounded if there exists  $b \in \mathbb{R}$  so that for all  $x \in I$ ,  $x \leq b$ . If  $\gamma$  is not bounded then  $\gamma$  is unbounded. A curve  $\delta : I_\delta \rightarrow M$  is a sub-curve of  $\gamma : I_\gamma \rightarrow M$  if  $I_\delta \subset I_\gamma$  and  $\gamma|_{I_\delta} = \delta$ . A change of parameter on  $\gamma : I \rightarrow M$  is a monotone increasing  $C^1$  function  $s : J \rightarrow I$ , where  $J \subset \mathbb{R}$  is an interval. Two curves  $\gamma : I_\gamma \rightarrow M$  and  $\delta : I_\delta \rightarrow M$  are related (or obtained) by a change of parameter if there exists a change of parameter  $s : I_\gamma \rightarrow I_\delta$  so that  $\delta \circ s = \gamma$ .

## 2. A coordinate independent chart induced boundary

This section presents a rigorous formulation of the currently heuristic use of charts to study global structure, as outlined in the introduction, and uses this to construct a coordinate independent chart induced boundary. Section 2.1 formalises what is meant by a chart ‘extending to the edge of a manifold’ and defines a class of such charts that are suitable for the use in the analysis of quantities defined ‘on the edge’. Section 2.2 defines an equivalence relation that is an extension of coordinate transformations between charts to the boundary. In Section 2.3 a topological space is constructed using this equivalence relation. This topological space is the completion of the manifold by the boundary structures induced by the charts that ‘extend to the edge’. The manifold is embedded as a dense open set into this topological space. A collection of homeomorphisms are also defined. On the image of the manifold these homeomorphisms are the normal coordinate transformations. On the topological boundary of the manifold in the topological space they can be thought of as ‘coordinate transformations on the boundary’. Section 2.4 studies what conditions are needed for the topological space to be a manifold with boundary and for the coordinate transformations on the boundary to be genuine coordinate transformations. Lastly, Section 2.5 presents a physically motivated classification of the elements of the topological boundary of the manifold in the topological space.

Note that the constructions below correspond, in a well defined way, to constructions used in the theory of Cauchy spaces. The study of such spaces gives a natural way to discuss compactifications and completions, [19, 3]. Hence it should be no surprise that the material of this paper is related to them. All of the material of this section, excluding the classification, can be rephrased in terms of Cauchy spaces. The references [27, 28] contain further discussion of this.

### 2.1. Boundaries induced by charts

This section identifies which points in the topological boundary of the range of a chart can be considered as representing a portion of the boundary of the manifold and which charts are suitable for use in this way. I show that every suitable chart induces an envelopment of the manifold so that the topological boundary given by the envelopment is the same as the boundary given by the chart. Thus each suitable chart also allows for calculations on the boundary that require differential structure.

**Definition 2.1.** *Let  $M$  be a manifold and  $\alpha \in \mathcal{A}(M)$  a chart. An admissible boundary point of  $\alpha$  is an element  $p \in \partial \text{ran}(\alpha)$  so that for all sequences  $(x_i) \subset \text{dom}(\alpha)$ , if  $\alpha(x_i) \rightarrow p$  then  $(x_i)$  has no accumulation points in  $M$ . The set of all admissible boundary points of  $\alpha$  will be denoted  $BP(\alpha)$ .*

The elements of  $BP(\alpha)$  can be thought of as those points on the ‘edge’ of  $M$  that  $\alpha$  extends to.



**Example 2.2.** Let  $M = \mathbb{R}^2$  and let  $\alpha : \mathbb{R}^2 \setminus \{(x, 0) : x \leq 0\} \rightarrow \mathbb{R}^2$  be the chart given by

$$\alpha(x, y) = \left( \frac{2}{\pi} \tan^{-1} \left( \sqrt{x^2 + y^2} \right), \arctan(x, y) \right),$$

where  $\arctan : \mathbb{R} \times \mathbb{R} \rightarrow (-\pi, \pi)$  is defined by

$$\arctan(x, y) = \begin{cases} \tan^{-1} \left( \frac{y}{x} \right) & x > 0 \\ \tan^{-1} \left( \frac{y}{x} \right) + \pi & x < 0, y > 0 \\ \tan^{-1} \left( \frac{y}{x} \right) - \pi & x < 0, y < 0 \\ \frac{\pi}{2} & x = 0, y > 0 \\ -\frac{\pi}{2} & x = 0, y < 0. \end{cases}$$

This chart can be thought of as ‘compactified polar coordinates’ on the plane. The range of  $\alpha$  is  $(0, 1) \times (-\pi, \pi)$ . The set  $\partial \text{ran}(\alpha)$  is the union of the four sets

$$\begin{aligned} S_1 &= \{1\} \times [-\pi, \pi], & S_2 &= \{0\} \times [-\pi, \pi], \\ S_3 &= (0, 1) \times \{-\pi\}, & S_4 &= (0, 1) \times \{\pi\}. \end{aligned}$$

A sequence  $(x_i) \subset \text{ran}(\alpha)$  with an endpoint in;

1.  $S_1$  is such that  $(\alpha^{-1}(x_i))$  has no accumulation points,
2.  $S_2$  is such that  $(\alpha^{-1}(x_i))$  has  $(0, 0)$  as an endpoint,
3.  $S_3$  or  $S_4$  is such that  $(\alpha^{-1}(x_i))$  has an accumulation point in  $\{(x, 0) : x < 0\}$ .

Hence  $BP(\alpha) = S_1$ .

Not every chart with admissible boundary points gives a suitable representation of the boundary.

**Example 2.3.** Let  $M = \mathbb{R}^2 \setminus \{(x, y) \in \mathbb{R}^2 : y \geq |x|\}$ . The chart  $\alpha : \{(x, y) \in \mathbb{R}^2 : y < 0\} \rightarrow \mathbb{R}^2$  given by  $\alpha(x, y) = (x, y)$  has the point  $(0, 0)$  as its only admissible boundary point.

Charts like  $\alpha$  of Example 2.3 describe a single isolated boundary point. Hence the chart is unable to describe topological, let alone differential, structure on the ‘edge’ of the manifold. The following definition describes the class of charts that avoid this issue.

**Definition 2.4.** Let  $M$  be a manifold,  $\alpha \in \mathcal{A}(M)$  and  $U \subset \mathbb{R}^n$ , an open set, so that  $\text{ran}(\alpha) \subset U$  and  $\emptyset \neq \partial_U \text{ran}(\alpha) \subset BP(\alpha)$ . Then the pair  $(\alpha, U)$  is an extension,  $\alpha$  is extendible and  $U$  is an extension of  $\alpha$ . Let

$$EX(M) = \{(\alpha, U) : \alpha \in \mathcal{A}(M), U \text{ is an extension of } \alpha\}$$

be the set of all extensions.



Propositions 2.8 and 2.9, below, justify the use of the word extension and demonstrate that  $\partial_U \text{ran}(\alpha)$  can be thought of as a representation of a portion of the boundary of  $M$ . Note that if  $(\alpha, U)$  is an extension then  $\partial_U \text{ran}(\alpha) = \text{BP}(\alpha) \cap U$  as both  $\text{ran}(\alpha)$  and  $U$  are open.

**Example 2.5.** Continuing from Example 2.2: let  $U = (0, \infty) \times (-\pi, \pi)$  then  $\text{ran}(\alpha) \subset U$  and  $\partial_U \text{ran}(\alpha) = S_1 = \text{BP}(\alpha)$ . Thus  $(\alpha, U)$  is an extension.

Continuing from Example 2.3: every neighbourhood of  $(0, 0)$  contains a point in  $\partial \text{ran}(\alpha)$  that is not in  $\text{BP}(\alpha)$ . Therefore  $\alpha$  has no extensions.

By allowing  $U$  to vary, flexibility to remove admissible boundary points has been introduced into the definition of an extension. This could be useful if, for some reason, a portion of  $\text{BP}(\alpha)$  was considered ‘artificial’ while some other portion was considered ‘natural’. Note, however that for any extendible chart there is a maximum extension.

**Lemma 2.6.** Let  $\alpha \in \mathcal{A}(M)$  be extendible. Then there exists a unique  $V \subset \mathbb{R}^n$  so that  $(\alpha, V) \in \text{EX}(M)$  and so that if  $(\alpha, U) \in \text{EX}(M)$  then  $U \subset V$ .

*Proof.* Let  $\mathcal{C} = \{U \subset \mathbb{R}^n : (\alpha, U) \in \text{EX}(M)\}$ . Let  $V = \bigcup \mathcal{C}$ . Since  $\alpha$  is extendible  $\mathcal{C} \neq \emptyset$  and hence  $V \neq \emptyset$ . Since  $\text{ran}(\alpha) \subset U$  for all  $U \in \mathcal{C}$ ,  $\text{ran}(\alpha) \subset V$ . Since  $\text{ran}(\alpha)$  and  $V$  are open  $\partial_V \text{ran}(\alpha) = \bigcup_{U \in \mathcal{C}} \partial_U \text{ran}(\alpha)$ . Thus  $\partial_V \text{ran}(\alpha) \subset \text{BP}(\alpha)$ . Lastly since  $\alpha$  is extendible there exists  $U \subset \mathbb{R}^n$  so that  $(\alpha, U) \in \text{EX}(M)$ . Thus  $\partial_U \text{ran}(\alpha) \neq \emptyset$  and so, from above,  $\partial_V \text{ran}(\alpha) \neq \emptyset$ . That is  $(\alpha, V) \in \text{EX}(M)$ . It is clear, by construction, that if  $(\alpha, U) \in \text{EX}(M)$  then  $U \subset V$ . This also implies the uniqueness of  $V$ . Hence we have the result.  $\square$

**Definition 2.7.** Let  $(\alpha, U) \in \text{EX}(M)$  and  $V \subset \text{BP}(\alpha) \cap U$ . The triple  $(\alpha, U, V)$ , or sometimes just  $V$ , is called a boundary set. If  $V = \{p\}$  then the triple  $(\alpha, U, \{p\})$ , or sometimes just  $p$  is called a boundary point. The set of all boundary points is

$$B_{ch}(M) = \{(\alpha, U, \{p\}) : (\alpha, U) \in \text{EX}(M), p \in \text{BP}(\alpha) \cap U\}.$$

The chart based approach to studying the ‘edge’ of a manifold chooses a chart,  $\alpha$ , and identifies some subset of  $\partial \text{ran}(\alpha)$  as representing the ‘edge’ of the manifold. In the framework above this subset is the set  $\partial_U \text{ran}(\alpha)$  associated to an extension  $(\alpha, U)$  of  $\alpha$ . In this way I have attempted to formalise the chart based approach.

I now turn to describing the envelopment induced by an extension. As mentioned above given an extension  $(\alpha, U) \in \text{EX}(M)$  the set  $\partial_U \text{ran}(\alpha) = \text{BP}(\alpha) \cap U$  can be thought of as a representation of a portion of the boundary of  $M$ . The following proposition makes this idea precise by showing how an extension induces an envelopment of the manifold so that the topological boundary of the image on the manifold is  $\text{BP}(\alpha) \cap U$ . The range of the envelopment is the topological pasting of  $M$  and  $U$  via the diffeomorphism  $\alpha$ , which is a manifold by the definition of an extension.

**Proposition 2.8.** Let  $(\alpha, U) \in \text{EX}(M)$  then there exists a manifold  $M_{(\alpha, U)}$  and an envelopment  $\Psi(\alpha, U) : M \rightarrow M_{(\alpha, U)}$  so that  $\partial \Psi(\alpha, U)(M) = \text{BP}(\alpha) \cap U$ .

*Proof.* Let  $(\alpha, U) \in \text{EX}(M)$  and let  $\alpha_U : \text{dom}(\alpha) \cup (U \setminus \text{ran}(\alpha)) \rightarrow U$  be defined by

$$\alpha_U(x) = \begin{cases} \alpha(x) & x \in \text{dom}(\alpha) \\ x & x \in U \setminus \text{ran}(\alpha). \end{cases}$$

Give the set  $M_{(\alpha, U)} = M \cup \{U \setminus \text{ran}(\alpha)\}$  the topology

$$\mathcal{T} = \{U \subset M : U \text{ is open in } M\} \cup \{\alpha_U^{-1}(V) \subset \text{dom}(\alpha) \cup (U \setminus \text{ran}(\alpha)) : V \text{ is open in } U\}.$$

The topological space  $M_{(\alpha, U)}$  equipped with the atlas generated by the functions  $\mathcal{A}(M) \cup \{\alpha_U\}$  is a manifold. The map  $\Psi(\alpha, U) : M \rightarrow M_{(\alpha, U)}$  given by  $\Psi(\alpha, U)(x) = x$  is the required envelopment. The proofs of these claims are straightforward, and follow directly from the definition of  $\mathcal{T}$ , the definition of  $\text{EX}(M)$  and as  $\alpha$  is a diffeomorphism. The details are left to the reader.

I now show that  $\partial\Psi(\alpha, U)(M) = \text{BP}(\alpha) \cap U$ . Let  $x \in \partial\Psi(\alpha, U)(M)$ . By construction this implies that  $x \in U \setminus \text{ran}(\alpha)$ . If  $x$  is in the interior of  $U \setminus \text{ran}(\alpha)$  then there exists  $V$  an open subset of  $U$  so that  $V \subset U \setminus \text{ran}(\alpha)$ . The definition of  $\mathcal{T}$  implies that  $x \notin \partial\Psi(\alpha, U)(M)$ . Thus  $x$  is not in the interior of  $U \setminus \text{ran}(\alpha)$ . Since  $U$  and  $\text{ran}(\alpha)$  are open this implies that  $x \in \partial_U \text{ran}(\alpha)$ . Thus there exists  $(x_i) \subset \text{dom}(\alpha)$  so that  $(\alpha(x_i)) \rightarrow x$ . The definition of  $\mathcal{T}$  implies that  $(\Psi(\alpha, U)(x_i)) \rightarrow x$ . Since  $M_{(\alpha, U)}$  is Hausdorff  $(\Psi(\alpha, U)(x_i))$  has no limit points in  $\Psi(\alpha, U)(M)$ . As  $\Psi(\alpha, U)$  is a homeomorphism this implies that  $(x_i)$  has no limit points in  $M$ . Therefore  $x \in \text{BP}(\alpha)$ . From above,  $x \in \partial_U \text{ran}(\alpha) \subset U$ . That is  $\partial\Psi(\alpha, U)(M) \subset \text{BP}(\alpha) \cap U$ .

Suppose that  $x \in \text{BP}(\alpha) \cap U$ . By definition there exists  $(x_i) \subset \text{dom}(\alpha)$  so that  $(\alpha(x_i)) \rightarrow x$  and  $(x_i)$  has no accumulation points in  $M$ . The definition of  $\mathcal{T}$  implies that  $(\Psi(\alpha, U)(x_i)) \rightarrow x$ . Hence  $x \in \overline{\Psi(\alpha, U)(M)}$ . If  $x \in \Psi(\alpha, U)(M)$  then, as  $\Psi(\alpha, U)$  is a homeomorphism,  $(x_i)$  would have  $\Psi(\alpha, U)^{-1}(x)$  as an accumulation point. This is a contradiction, however. Hence  $x \in \partial\Psi(\alpha, U)(M)$  and therefore  $\text{BP}(\alpha) \cap U \subset \partial\Psi(\alpha, U)(M)$ .

Thus  $\partial\Psi(\alpha, U)(M) = \text{BP}(\alpha) \cap U$  as required.  $\square$

There is a converse to the above Proposition.

**Proposition 2.9.** *Let  $\phi : M \rightarrow M_\phi$  be an envelopment and let  $\alpha \in \mathcal{A}(M_\phi)$ . If  $\text{dom}(\alpha) \cap \partial\phi(M) \neq \emptyset$  then  $(\alpha \circ \phi, \text{ran}(\alpha)) \in \text{EX}(M)$ , where  $\alpha \circ \phi$  denotes the map  $\alpha \circ \phi = \alpha \circ \phi|_{\phi^{-1}(\text{dom}(\alpha) \cap \phi(M))}$ .*

*Proof.* It is clear that  $\text{ran}(\alpha \circ \phi) \subset \text{ran}(\alpha)$ . Since  $\text{dom}(\alpha) \cap \partial\phi(M) \neq \emptyset$  we know that  $\partial_{\text{ran}(\alpha)} \text{ran}(\alpha \circ \phi) \neq \emptyset$ . Let  $x \in \partial_{\text{ran}(\alpha)} \text{ran}(\alpha \circ \phi)$ . Let  $(x_i) \subset \text{dom}(\alpha \circ \phi)$  be a sequence so that  $\alpha \circ \phi(x_i) \rightarrow x$ . Since  $\alpha$  is a diffeomorphism  $\phi(x_i) \rightarrow \alpha^{-1}(x)$ . By assumption  $x \in \partial_{\text{ran}(\alpha)} \text{ran}(\alpha \circ \phi)$  which implies that  $\alpha^{-1}(x) \in \partial\phi(M)$ . Since  $\phi$  is a diffeomorphism and  $M_\phi$  is Hausdorff this implies that the sequence  $(x_i)$  has no limit points in  $M$ . Therefore  $x \in \text{BP}(\alpha \circ \phi)$ . Hence  $\partial_{\text{ran}(\alpha)} \text{ran}(\alpha \circ \phi) \subset \text{BP}(\alpha \circ \phi)$  as required.  $\square$

These two results show that, for any extension  $(\alpha, U)$  it makes sense to think of  $\text{BP}(\alpha) \cap U$  as the representation of a portion of the boundary of  $M$  given by the chart  $\alpha$ . This thought can only be taken so far, however, as there is no guarantee that  $\overline{\Psi(\alpha, U)(M)}$  is a manifold with boundary.

**Example 2.10.** Let  $M = \{(x, y) : y < \sin(\frac{1}{x}), x > 0\}$  and define the chart  $\alpha : M \rightarrow \mathbb{R}^2$  by  $\alpha(x, y) = (x, y)$ . Let  $U = \mathbb{R}^2$  then  $(\alpha, U) \in \text{EX}(M)$  and  $M_{(\alpha, U)} = \mathbb{R}^2$ . The set  $\overline{\Psi(\alpha, M)(M)}$  is

$$M \cup \{(0, y) : y < -1\} \cup \overline{\left\{ \left( x, \sin\left(\frac{1}{x}\right) \right) : x > 0 \right\}}.$$

The last set in the union contains points,  $\{(0, y) \in \mathbb{R}^2 : -1 < y \leq 1\}$ , that are not the endpoint of any curve in  $M$ . As a consequence  $\overline{\Psi(\alpha, U)(M)}$  is not a manifold with boundary. The space  $\overline{\Psi(\alpha, U)(M)}$  must be considered as a topological space which has an open dense subset which is a manifold.

## 2.2. Relationships between boundaries given by extensions

The boundary  $\text{BP}(\alpha) \cap U$  of an extension  $(\alpha, U)$  is, naturally, dependent on  $(\alpha, U)$  and thus is not chart independent. Of course the same is true of  $\text{ran}(\alpha)$ . This is a chart dependent way of viewing the manifold. In the case of  $\text{ran}(\alpha)$  the solution to chart dependence is a part of the basic apparatus of differential geometry: quantities defined on  $\text{ran}(\alpha)$  are well defined on  $M$  if they transform the ‘right way’ under changes of coordinates.

In the context of this paper, where I am concerned with a construction of a topological space (the boundary of  $M$ ), transforming the right way means that  $x \in \text{ran}(\alpha)$  represents the same point in the manifold as  $y \in \text{ran}(\beta)$  if and only if  $\beta \circ \alpha^{-1}(x) = y$ . Let  $(\alpha, U)$  and  $(\beta, X)$  be extensions. If the function  $\beta \circ \alpha^{-1} : \alpha(\text{dom}(\alpha) \cap \text{dom}(\beta)) \rightarrow \beta(\text{dom}(\alpha) \cap \text{dom}(\beta))$  could be extended to some portion of  $\partial_U \text{ran}(\alpha)$  and  $\partial_X \text{ran}(\beta)$  then this extended function would give a natural way to think about coordinate invariance on the boundary of the manifold. The example below shows that it is not always possible to extend  $\beta \circ \alpha^{-1}$  to all of  $\partial_U \text{ran}(\alpha)$  as a function. It is, however, always possible to extend it as a relation.

Each point in  $\text{ran}(\alpha) \cup \partial_U \text{ran}(\alpha)$  can be uniquely equated with the set of sequences in  $\text{ran}(\alpha)$  that limit to the point. Since each sequence is in  $\text{ran}(\alpha)$  we can apply  $\beta \circ \alpha^{-1}$  to it. If the image of the sequence has a limit point, then the relation extending  $\beta \circ \alpha^{-1}$  will relate the point in  $\text{ran}(\alpha) \cup \partial_U \text{ran}(\alpha)$  and the limit point. This relation can be thought of as the ‘coordinate transform on the edge of the manifold’ between  $\alpha$  and  $\beta$ , though in Proposition 2.17 I define functions which are better suited to this terminology.

Because of the use of a relation and not a function it is possible for a boundary point of some extension to correspond to a boundary set of another. This is a down side to the construction, which can be addressed through clever choices of charts, see Section 2.4. The example below, which illustrates this behaviour, is an expansion of a brief discussion given in Geroch’s 1968 paper [17].

After the example the remainder of this section will define the relation, alluded to above, and discuss when a boundary point/set should correspond to a boundary point/set in some other extension.

**Example 2.11** ([16]). Let  $h : [0, \pi) \rightarrow [0, 2)$  be the smooth function defined by

$$h(\theta) = \begin{cases} 0 & \theta = 0 \\ \frac{1}{1 + \exp\left(\frac{\pi}{4}\left(\frac{1}{\theta} - \frac{4}{\pi - 4\theta}\right)\right)} & \theta \in \left(0, \frac{\pi}{4}\right) \\ 1 & \theta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right] \\ 1 + \frac{1}{1 + \exp\left(\frac{\pi}{4}\left(\frac{1}{\theta - \frac{3\pi}{4}} - \frac{4}{\pi - 4(\theta - \frac{3\pi}{4})}\right)\right)} & \theta \in \left(\frac{3\pi}{4}, \pi\right) \end{cases}$$

and let  $f : [0, 1) \times [0, \pi) \rightarrow [0, 1) \times [0, \pi)$  be defined by  $f(r, \theta) = (r, \theta(1 - r) + r\frac{\pi}{2}h(\theta))$ . The function  $f$  can be thought of as a homotopy between the constant function  $\theta \mapsto \theta$  and a function which maps  $[0, \pi)$  onto itself while contracting  $[\frac{\pi}{4}, \frac{3\pi}{4}]$  to the point  $\frac{\pi}{2}$ . The function  $f$  is a smooth bijection hence its inverse,  $f^{-1}$ , is well defined.

Let  $M = \{(x, y) : x^2 + y^2 < 1\}$  and define the following charts  $\alpha : M \rightarrow \mathbb{R}^2$  and  $\beta : M \rightarrow \mathbb{R}^2$  given by, in polar coordinates,  $\alpha(r, \theta) = (r, \theta)$  and

$$\beta(r, \theta) = \begin{cases} f(r, \theta) & \theta \in [0, \pi) \\ f^{-1}(r, \theta - \pi) + (0, \pi) & \theta \in [\pi, 2\pi). \end{cases}$$

Then the transition function  $\phi = \beta \circ \alpha^{-1}$  is given by  $\phi = \beta$ . The set  $\mathbb{R}^2$  extends both  $\alpha$  and  $\beta$ .

For each  $i \in \mathbb{N}$ , let

$$u_i = \left(\frac{i}{i+1}, \frac{5\pi}{4} + \frac{\pi}{4} \frac{i}{i+1}\right), \quad v_i = \left(\frac{i}{i+1}, \frac{7\pi}{4} - \frac{\pi}{4} \frac{i}{i+1}\right).$$

Note that these sequences have the following limits

$$u_i \rightarrow \left(1, \frac{3\pi}{2}\right), \quad v_i \rightarrow \left(1, \frac{3\pi}{2}\right).$$

The images of these sequences under  $\phi$  are

$$\phi(u_i) = \left(\frac{i}{i+1}, \frac{5\pi}{4}\right), \quad \phi(v_i) = \left(\frac{i}{i+1}, \frac{7\pi}{4}\right).$$

The limits of these images are

$$\phi(u_i) \rightarrow \left(1, \frac{5\pi}{4}\right), \quad \phi(v_i) \rightarrow \left(1, \frac{7\pi}{4}\right).$$

Hence while  $u_i, v_i$  identify the same boundary point in  $(\alpha, \mathbb{R}^2)$  they identify different boundary points in  $(\beta, \mathbb{R}^2)$ . This implies that there is no continuous extension of  $\phi$  to a neighbourhood of  $(1, \frac{3\pi}{2})$ . In accordance with the discussion above, the boundary point  $(1, \frac{3\pi}{2})$  of  $(\alpha, \mathbb{R}^2)$  is considered to be represented by a set of boundary points with respect to  $(\beta, \mathbb{R}^2)$  which contains the points  $(1, \frac{5\pi}{4})$  and  $(1, \frac{7\pi}{4})$ . It is possible to explicitly identify this set.

Let  $X = \{(1, \theta) : \theta \in [\frac{5\pi}{4}, \frac{7\pi}{4}]\}$ . Similar calculations, as above, show that every sequence  $(x_i) \subset \text{ran}(\beta)$  that converges to a point in  $X$  is such that  $\phi^{-1}(x_i) \rightarrow (1, \frac{3\pi}{2})$ . As well as that every sequence,  $(x_i) \subset \text{ran}(\alpha)$  that converges to  $(1, \frac{3\pi}{2})$  is such that the accumulation points of the sequence  $(\phi(x_i))$  lie in  $X$ . Hence, considering boundary points to be sets of sequences we should consider  $(1, \frac{3\pi}{2})$  and  $X$  as different representations of the same boundary point with respect to the extensions  $(\alpha, \mathbb{R}^2)$  and  $(\beta, \mathbb{R}^2)$ .

Now consider an element  $x \in X$ . From the discussion above it is clear that if  $(x_i) \subset \text{ran}(\beta)$  is such that  $x_i \rightarrow x$  then  $\phi^{-1}(x_i) \rightarrow (1, \frac{3\pi}{2})$ . Thus the set of sequences which identify  $x$ , under  $\phi^{-1}$ , can be viewed as a subset of the set of sequences which identify  $(1, \frac{3\pi}{2})$ .

The same arguments can be used to show that the boundary point  $(1, \frac{\pi}{2})$  of  $(\beta, \mathbb{R}^2)$  and the set of boundary points  $\{(1, \theta) : \theta \in [\frac{\pi}{4}, \frac{3\pi}{4}]\}$  of  $(\alpha, \mathbb{R}^2)$  should be considered as different representations of the same portion of the boundary point with respect to the extensions  $(\beta, \mathbb{R}^2)$  and  $(\alpha, \mathbb{R}^2)$ .

The definition of the relation, mentioned above, is as follows.

**Definition 2.12.** The boundary set  $(\alpha, U, V)$  covers the boundary set  $(\beta, X, Y)$ ,  $(\alpha, U, V) \triangleright (\beta, X, Y)$ , if for all  $(y_i) \subset \text{dom}(\beta)$  so that  $(\beta(y_i))$  has an accumulation point in  $Y$ , there exists a subsequence  $(v_i)$  of  $(y_i)$  so that  $(v_i) \subset \text{dom}(\alpha)$  and  $(\alpha(v_i))$  has an accumulation point in  $V$ .

Hence if  $(\alpha, U, V) \triangleright (\beta, X, Y)$  then the map  $\alpha \circ \beta^{-1}$  can be considered as mapping  $Y$  into  $V$ .

**Definition 2.13.** The boundary sets  $(\alpha, U, V)$  and  $(\beta, X, Y)$  are equivalent,  $(\alpha, U, V) \equiv (\beta, X, Y)$  if and only if  $(\alpha, U, V) \triangleright (\beta, X, Y)$  and  $(\beta, X, Y) \triangleright (\alpha, U, V)$ . The equivalence class of  $(\alpha, U, V)$  is denoted  $[(\alpha, U, V)]$ .

The pre-order  $\triangleright$  induces a partial order, which in an abuse of notation is also denoted by  $\triangleright$ , on the set of all equivalence classes: the equivalence class of boundary sets  $[(\alpha, U, V)]$  covers the equivalence class of boundary sets  $[(\beta, X, Y)]$ ,  $[(\alpha, U, V)] \triangleright [(\beta, X, Y)]$  if and only if  $(\alpha, U, V) \triangleright (\beta, X, Y)$ .

**Example 2.14.** Continuing from Example 2.11: the boundary point  $(\alpha, \mathbb{R}^2, \{(1, \frac{3\pi}{2})\})$  is equivalent to the boundary set  $(\beta, \mathbb{R}^2, X)$  and for any  $x \in X$  it covers the boundary point  $(\beta, \mathbb{R}^2, \{x\})$ . Similarly, the boundary point  $(\beta, \mathbb{R}^2, \{(1, \frac{\pi}{2})\})$  is equivalent to the boundary set  $(\alpha, \mathbb{R}^2, \{(1, \theta) : \theta \in [\frac{\pi}{4}, \frac{3\pi}{4}]\})$  and for point  $y$  in  $\{(1, \theta) : \theta \in [\frac{\pi}{4}, \frac{3\pi}{4}]\}$  it covers  $(\alpha, \mathbb{R}^2, \{y\})$ .

If  $(\alpha, U) \in \text{EX}(M)$  and  $x \in \text{BP}(\alpha) \cap U$ , then  $(\alpha, U, \{x\})$  is a representation of  $[(\alpha, U, \{x\})]$ . The equivalence class  $[(\alpha, U, \{x\})]$  can be thought of as the coordinate independent object that  $\alpha$  provides a representation of.

### 2.3. The chart induced coordinate invariant boundary

I am now in a position to construct the space which each extension  $(\alpha, U)$  gives a representation of.

**Definition 2.15.** Let  $Q \subset EX(M)$  be a set of extensions and let  $P_Q = \{(\alpha, \text{ran}(\alpha)) : \alpha \in \mathcal{A}(M)\}$ . While no element of  $P_Q$  is an extension (as  $\emptyset = \partial_{\text{ran}(\alpha)} \text{ran}(\alpha)$ ), a pair  $(\alpha, \text{ran}(\alpha)) \in P_Q$  can be viewed as the trivial extension of  $\alpha$ . Let  $S_Q = P_Q \cup Q$ . For each pair  $(\alpha, U) \in S_Q$  let  $N_{(\alpha, U)} = \text{ran}(\alpha) \cup \partial_U \text{ran}(\alpha)$ . Each  $N_{(\alpha, U)} \subset \mathbb{R}^n$  and thus carries the relative topology induced by the topology on  $\mathbb{R}^n$ . Let

$$N_Q = \bigsqcup_{(\alpha, U) \in S_Q} N_{(\alpha, U)}.$$

be the disjoint of all  $N_{(\alpha, U)}$ . Define an equivalence relation on  $N_Q$  by  $x \in N_{(\alpha, U)}$  is equivalent to  $y \in N_{(\beta, X)}$  if and only if either  $x \in \text{ran}(\alpha), y \in \text{ran}(\beta)$  and  $\beta \circ \alpha^{-1}(x) = y$  or  $x \in \partial_U \text{ran}(\alpha), y \in \partial_X \text{ran}(\beta)$  and  $(\alpha, U, \{x\}) \equiv (\beta, X, \{y\})$ . In an abuse of notion, let  $Q(M)$  be the image of  $N_Q$  under this equivalence relation. There exists a quotient map  $q : N_Q \rightarrow Q(M)$  that takes each element  $x \in N_Q$  to its equivalence class  $q(x) = [x]$ . The set  $N_Q$  has a natural topology induced by the topologies on each of its components. Equip  $Q(M)$  with the quotient topology induced by the topology on  $N_Q$  and the map  $q$ . Hence  $Q(M)$  is a topological space. The set  $Q(M)$  will be called the completion of  $M$  with respect to  $Q$ . There exists a map  $\iota_Q : M \rightarrow Q(M)$  given by  $\iota_Q(x) = [\alpha(x)]$  for any  $\alpha \in \mathcal{A}(M)$  so that  $x \in \text{dom}(\alpha)$ . The map  $\iota_Q$  is well defined by definition of  $Q(M)$ .

The properties of  $q, \iota_Q$  and  $Q(M)$  are described in Appendix A. For the convenience of the reader I summarise them here. The map  $q$  is open and continuous and the restriction  $q|_{N_{(\alpha, U)}}$ , for all  $(\alpha, U) \in S_Q$ , is a homeomorphism. The map  $\iota_Q$  is continuous and a homeomorphism onto its image. The space  $Q(M)$  is a  $T_1$ , separable, first countable, locally metrizable topological space. Lastly  $\iota_Q(M)$  is an open dense subset of  $Q(M)$ .

The boundary  $\partial_{\iota_Q(M)}$  is what you might expect it to be from the discussion in Section 2.1.

**Proposition 2.16.** Let  $Q \subset EX(M)$  and let  $\iota_Q$  be the function defined above. Then

$$\partial_{\iota_Q(M)} = \bigcup_{(\alpha, U) \in Q} q(\partial_U \text{ran}(\alpha)) = \bigcup_{(\alpha, U)} q(BP(\alpha) \cap U).$$

*Proof.* The second equality follows by definition of  $EX(M)$ , Definition 2.4.

Suppose that there exists  $(\alpha, U) \in Q$  and  $x \in \text{ran}(\alpha)$  so that  $q(x) \in \partial_{\iota_Q(M)}$ . By assumption  $\alpha^{-1}(x) \in M$  and  $\iota_Q(\alpha^{-1}(x)) = q(x)$ . Since  $\iota_Q$  is a homeomorphism, Proposition A.6, and  $M$  is open this implies that  $q(x) \notin \partial_{\iota_Q(M)}$ , a contradiction.

Suppose that there exists  $(\alpha, U) \in Q$  and  $x \in \partial_U \text{ran}(\alpha)$  so that  $q(x) \notin \partial_{\iota_Q(M)}$ . By Proposition A.6  $Q(M) = \iota_Q(M) \cup \partial_{\iota_Q(M)}$  thus  $q(x) \in \iota_Q(M)$ . As  $\iota_Q$  is a homeomorphism onto its image this implies that there exists  $y \in M$  so that  $q(y) = q(x)$ . This implies that  $x \notin BP(\alpha)$ . This is a contradiction since  $(\alpha, U) \in EX(M)$ .  $\square$

Since  $Q(M)$  is intended to be a completion of  $M$  a full investigation of the compactness properties of  $Q(M)$  would be interesting. Unfortunately the, in general, non-Hausdorff behaviour of points in  $Q(M)$  make such a study a lengthy and technical affair, thus placing it beyond the scope of this paper. One can prove, however, that  $Q(M)$  is a completion of  $M$ , in the sense of [19, Definition 1.4.2], once  $M$  is equipped with the Cauchy structure induced by all sequences in  $M$  whose images, under some chart in  $Q$ , are convergent. For the purposes of this paper, and most applications, Proposition 2.21 is sufficient.

The space  $Q(M)$  does not, in general, carry a differential structure and, in general, is not locally homeomorphic to  $\mathbb{R}^{n-1} \times \{x \geq 0 : x \in \mathbb{R}\}$ . However, the functions  $q|_{N_{(\alpha,U)}} : N_{(\alpha,U)} \rightarrow q(N_{(\alpha,U)})$  are homeomorphisms onto their images in  $Q(M)$ . By Proposition 2.8 the set  $\alpha_U^{-1}(N_{(\alpha,U)})$  is a subset of  $M_{(\alpha,U)}$ . Thus  $Q(M)$  is locally homeomorphic to suitable subspaces of manifolds. What prevents  $Q(M)$  from being a, in general, non-second countable, non-Hausdorff manifold is that the maps  $q|_{N_{(\alpha,U)}} : N_{(\alpha,U)} \rightarrow q(N_{(\alpha,U)})$  are not charts because, in general,  $N_{(\alpha,U)}$  is not an open subset of  $\mathbb{R}^{n-1} \times \{x \geq 0 : x \in \mathbb{R}\}$ . Never-the-less each  $N_{(\alpha,U)} \subset U$  and so as  $U = \text{ran}(\alpha_U)$ , each  $N_{(\alpha,U)}$  can be considered as a chart-like structure on  $Q(M)$  that carries a differential structure induced by  $\alpha$ . In Section 2.4 conditions under which  $Q(M)$  is a manifold with boundary are presented. In these circumstances  $q|_{N_{(\alpha,U)}}$  is a diffeomorphism and hence the atlas of  $Q(M)$  is generated by the maps  $q|_{N_{(\alpha,U)}}$ .

Since Definition 2.15 and the results of Appendix A only require a subset of extensions they can be used to, for example, define the  $C^1$  boundary of  $M$ , by requiring that each selected extension,  $(\alpha, U)$  to be such that  $\partial_U \text{ran}(\alpha)$  is  $C^1$  in  $\mathbb{R}^n$ .

Since  $Q(M)$  is composed of equivalence classes under  $\equiv$  the points in  $Q(M)$  are manifestly coordinate independent, in the sense that two boundary points  $(\alpha, U, \{x\})$  and  $(\beta, V, \{y\})$  are representations of the same third, coordinate independent, point  $[(\nu, W, \{z\})] \in Q(M)$  if and only if  $(\alpha, U, \{x\}) \equiv (\beta, V, \{y\})$ . This is a generalisation of the coordinate independence described in the beginning of Section 2.2. The condition  $(\alpha, U, \{x\}) \equiv (\beta, V, \{y\})$  can be rewritten as  $q(x) = q(y)$ , or perhaps even more evocatively as  $y = \left(q|_{N_{(\beta,V)}}\right)^{-1} \circ q|_{N_{(\alpha,U)}}(x)$ . The function  $\left(q|_{N_{(\beta,V)}}\right)^{-1} \circ q|_{N_{(\alpha,U)}}$  restricted to  $\alpha(\text{dom}(\alpha) \cap \text{dom}(\beta))$  is  $\beta \circ \alpha^{-1}$ . Hence the function  $\left(q|_{N_{(\beta,V)}}\right)^{-1} \circ q|_{N_{(\alpha,U)}}$  is an extension of the coordinate transformation  $\beta \circ \alpha^{-1}$  to the boundary of  $M$ . Functions of this form are the promised “coordinate transformations on the boundary”.

**Proposition 2.17.** *Let  $Q \subset EX(M)$  be a set of extensions. For every pair  $(\alpha, U), (\beta, V) \in S_Q$ , there exists a homeomorphism*

$$\begin{aligned} \overline{\beta \circ \alpha^{-1}} : q^{-1}(q(N_{(\alpha,U)}) \cap q(N_{(\beta,V)})) \cap N_{(\alpha,U)} \\ \rightarrow q^{-1}(q(N_{(\alpha,U)}) \cap q(N_{(\beta,V)})) \cap N_{(\beta,U)} \end{aligned}$$

given by

$$\overline{\beta \circ \alpha^{-1}}(x) = \left(q|_{N_{(\beta,V)}}\right)^{-1} \circ q|_{N_{(\alpha,U)}}(x),$$



so that  $\overline{\beta \circ \alpha^{-1}}|_{\alpha(\text{dom}(\alpha) \cap \text{dom}(\beta))} = \beta \circ \alpha^{-1}$  and  $\overline{\beta \circ \alpha^{-1}}(x) = y$  if and only if  $q(x) = q(y)$  and  $x \in N_{(\alpha, U)}$ ,  $y \in N_{(\beta, V)}$ , i.e. either  $\beta \circ \alpha^{-1}(x) = y$  or  $(\alpha, U, \{x\}) \equiv (\beta, V, \{y\})$ .

*Proof.* By Proposition A.4, for all  $(\alpha, U) \in S_Q$ ,  $q|_{N_{(\alpha, U)}}$  is a homeomorphism. Hence  $\overline{\beta \circ \alpha^{-1}}$  is a homeomorphism. Suppose that  $\overline{\beta \circ \alpha^{-1}}(x) = y$ . Then, by definition and as  $x \in N_{(\alpha, U)}$ ,  $y \in N_{(\beta, V)}$   $y = \left(q|_{N_{(\beta, V)}}\right)^{-1} \circ q|_{N_{(\alpha, U)}}(x)$  implies that  $q(y) = q(x)$  as required.

Similarly, if  $x \in N_{(\alpha, U)}$ ,  $y \in N_{(\beta, V)}$  and  $q(x) = q(y)$  then  $y = \left(q|_{N_{(\beta, V)}}\right)^{-1} \circ q|_{N_{(\alpha, U)}}(x)$  by definition of  $q$ .

I now show that  $\overline{\beta \circ \alpha^{-1}}|_{\alpha(\text{dom}(\alpha) \cap \text{dom}(\beta))} = \beta \circ \alpha^{-1}$ . By definition of  $q$ , if  $x \in \alpha(\text{dom}(\alpha) \cap \text{dom}(\beta))$  then there exists  $y \in \beta(\text{dom}(\alpha) \cap \text{dom}(\beta))$  so that  $\beta \circ \alpha^{-1}(x) = y$ . This implies that  $q(x) = q(y)$ . Since  $x \in N_{(\alpha, U)}$  and  $y \in N_{(\beta, V)}$  this implies that  $q(x) \in q(N_{(\alpha, U)}) \cap q(N_{(\beta, V)})$  and therefore that  $x$  is in the domain of  $\overline{\beta \circ \alpha^{-1}}$ . That is,  $\alpha(\text{dom}(\alpha) \cap \text{dom}(\beta)) \subset q^{-1}(q(N_{(\alpha, U)}) \cap q(N_{(\beta, V)}))$ . Thus  $\overline{\beta \circ \alpha^{-1}}$  is defined on  $\alpha(\text{dom}(\alpha) \cap \text{dom}(\beta))$ . The definition of  $q$  now implies that  $\overline{\beta \circ \alpha^{-1}}|_{\alpha(\text{dom}(\alpha) \cap \text{dom}(\beta))} = \beta \circ \alpha^{-1}$ , as required.  $\square$

The homeomorphism  $\overline{\beta \circ \alpha^{-1}}$ , due to its definition, can also be thought of as a map which describes the relation  $\equiv$  between  $(\alpha, U)$  and  $(\beta, V)$  restricted to boundary points and which respects the topologies given by  $U$  and  $V$ .

As already mentioned Proposition 2.17 allows the set  $Q$  to be free. The next result describes how the boundaries induced by different choices of sets of extensions are related. Note that this is not the most general form of the result possible, but it is sufficient for our needs.

**Proposition 2.18.** *Let  $A, B \subset EX(M)$  so that for all  $(\alpha, U) \in A$  there exists  $(\alpha, V) \in B$  with  $U \subset V$ . Then there is an injective continuous open function  $f : A(M) \rightarrow B(M)$  defined by  $f([x]) = [x]$ .*

*Proof.* Let  $(\alpha, U) \in A$  and let  $i : N_{(\alpha, U)} \rightarrow N_{(\alpha, V)}$  be the induced inclusion. The function  $f$  can alternatively be given by  $f|_{q(N_{(\alpha, U)})}([x]) = q|_{N_{(\alpha, V)}} \circ i \circ q|_{N_{(\alpha, U)}}^{-1}([x])$ . The claimed properties are now easy to check.  $\square$

We now give the coordinate independent chart induced boundary, which is the largest boundary with respect to the ordering of sub-boundaries given by Proposition 2.18.

**Definition 2.19** (The coordinate independent chart induced boundary). *For each  $\alpha \in \mathcal{A}(M)$  that is extendible let  $U_\alpha$  be the maximal extension given by Lemma 2.6. Let  $Q_{\mathcal{A}} = \{(\alpha, U_\alpha) : \alpha \in \mathcal{A}(M) \text{ is extendible}\}$ . Let  $Q_{\mathcal{A}}(M)$  and  $\iota_{Q_{\mathcal{A}}} : M \rightarrow Q_{\mathcal{A}}(M)$  be as given in Definition 2.15. The boundary of  $M$  is  $\partial \iota_{Q_{\mathcal{A}}}(M)$ . To emphasise the existence of the boundary independently from  $\iota_{Q_{\mathcal{A}}}$  I denote  $\partial \iota_{Q_{\mathcal{A}}}(M)$  by  $\mathcal{B}_{ch}(M)$ . The set  $Q_{\mathcal{A}}(M)$  will be called the completion of  $M$ .*

**Definition 2.20.** *In an abuse of notation, the elements of  $\mathcal{B}_{ch}(M)$  are called boundary points. The distinction between a boundary point  $[(\alpha, W, \{p\})] \in \mathcal{B}_{ch}(M)$  and a boundary point  $(\alpha, W, \{p\}) \in B_{ch}(M)$  will be clear from context.*

Due to the structure of  $Q_{\mathcal{A}}(M)$ , quantities defined using the standard chart based approach to studying the boundary of a manifold will be well defined on the completion of  $M$  if they respect the equivalence relation used to define  $Q_{\mathcal{A}}(M)$ . On  $\iota_{Q_{\mathcal{A}}}(M)$  this is nothing other than the usual invariance under changes of coordinates. On  $\mathcal{B}_{\text{ch}}(M)$  this can be interpreted as invariance under coordinate transformations on the boundary, as given by Proposition 2.17.

The space  $Q_{\mathcal{A}}(M)$  is a completion of  $M$  in the following sense.

**Proposition 2.21.** *Let  $Q_{\mathcal{A}}(M)$  be the completion of  $M$ , as given in Definition 2.19. If  $(x_i) \subset M$  is a sequence then  $(\iota_{Q_{\mathcal{A}}}(x_i))$  has at least one limit point.*

*Proof.* If  $x \in M$  is a limit point of  $(x_i)$  then as  $\iota_{Q_{\mathcal{A}}}$  is continuous, and  $Q_{\mathcal{A}}(M)$  is first countable,  $\iota_{Q_{\mathcal{A}}}(x)$  is a limit point of  $\iota_{Q_{\mathcal{A}}}(x_i)$ . If the sequence has no limit points then the Endpoint Theorem, [27, Theorem 3.2.1], shows that there exists an envelopment  $\phi : M \rightarrow M_\phi$  and  $x \in \partial\phi(M)$  so that  $\phi(x_i) \rightarrow x$ . Choose  $\beta \in \mathcal{A}(M_\phi)$  so that  $x \in \text{dom}(\beta)$ . Proposition 2.9 implies that  $(\beta \circ \phi, \text{ran}(\beta)) \in \text{EX}(M)$ . By construction  $(\beta \circ \phi, V) \in Q_{\mathcal{A}}$  where  $V$  is the set given by Lemma 2.6. By construction  $(\beta \circ \phi(x_i)) \subset N_{(\beta \circ \phi, \text{ran}(\beta))}$ ,  $\beta(x) \in N_{(\beta \circ \phi, \text{ran}(\beta))}$  and  $\beta \circ \phi(x_i) \rightarrow \beta(x)$  with respect to the topology on  $N_{(\beta \circ \phi, \text{ran}(\beta))}$ . Thus, as  $q$  is continuous and  $Q_{\mathcal{A}}(M)$  is first countable,  $\iota_{Q_{\mathcal{A}}}(x_i) = q(\beta \circ \phi(x_i)) \rightarrow q(\beta(x))$ . Since  $\iota_{Q_{\mathcal{A}}}(x_i) = q(\beta(x_i))$  the sequence  $(\iota_{Q_{\mathcal{A}}}(x_i))$  has  $q(\beta(x))$  as a limit point.  $\square$

As a consequence of Proposition 2.18, for any arbitrary  $A \subset \text{EX}(M)$ , the boundary  $\partial\iota_A(M)$  can be viewed as a subset of  $\mathcal{B}_{\text{ch}}(M)$ .

**Definition 2.22.** *Let  $A \subset \text{EX}(M)$  and let  $Q_A \subset \text{EX}(M)$  be as given in Definition 2.19. Let  $f : A(M) \rightarrow Q_A(M)$  be as given in Proposition 2.18. Let  $\sigma_A$  denote the set  $f(\partial\iota_A(M))$ . This is the set of boundary points induced by  $A$ . The topological space  $A(M)$  can be viewed as a concrete realisation of the sub-boundary  $\sigma_A$ . I shall sometimes refer to  $\sigma_A$  as the boundary induced by  $A$ . If for all sequences  $(x_i) \subset M$  the sequence  $(\iota_{Q_A}(x_i))$  has at least one limit point in  $f(A(M))$  then the set  $\sigma_A$  is called complete. If  $A = \{(\alpha, U)\}$  the set  $\sigma_A$  will be denoted by  $\sigma_{(\alpha, U)}$ . If the set  $U$  is clear from context  $\sigma_\alpha$  will be used.*

There is an important realisation of  $\mathcal{B}_{\text{ch}}(M)$  as a point set.

**Proposition 2.23.** *The boundary  $\mathcal{B}_{\text{ch}}(M)$  is in bijective correspondence with  $\frac{B_{\text{ch}}(M)}{\equiv}$ .*

*Proof.* This follows directly from the definitions of  $B_{\text{ch}}(M)$  (Definition 2.7),  $Q_{\mathcal{A}}(M)$  (Definition 2.19),  $\iota_{Q_{\mathcal{A}}}$  (Definition 2.15) and  $\equiv$  (Definition 2.13).  $\square$

Those familiar with the Abstract Boundary, [25], should note that  $\mathcal{B}_{\text{ch}}(M)$  has a similar structure. I exploit this similarity in Section 2.5 to give a physically motivated classification for the elements of  $\mathcal{B}_{\text{ch}}(M)$ . Never-the-less  $\mathcal{B}_{\text{ch}}(M)$  has a richer structure. In particular,  $\mathcal{B}_{\text{ch}}(M)$  comes as part of a topological space  $Q_{\mathcal{A}}(M)$  into which  $M$  is embedded, as  $\iota_{Q_{\mathcal{A}}}(M)$ , as an open dense set and the equivalence relation finds expression on  $Q_{\mathcal{A}}(M)$  as the coordinate transformations on the boundary which take the form of the homeomorphisms  $\overline{\beta \circ \alpha^{-1}}$ . These structures are not present in the Abstract Boundary. Refer to [2] for an example of a topology on the Abstract Boundary.

## 2.4. Boundaries induced by collections of compatible charts

In an ideal world a coordinate transformation on the boundary, Proposition 2.17, would actually be a coordinate transformation in some larger manifold. This section introduces a compatibility condition which is almost, but not quite enough to achieve this. If, in addition to the compatibility condition, the chosen set of extensions,  $Q$ , is countable and the extensions are sufficiently nice, see Proposition 2.27, then the coordinate transformations on the boundary turn out to be genuine coordinate transformations. The restrictions are mild and likely to be satisfied when performing computations. Thus, in this way, calculations on  $\mathcal{B}_{\text{ch}}(M)$ , or some suitable sub-boundary, are facilitated by the compatibility condition.

**Definition 2.24.** *The boundary points  $(\alpha, U, \{p\})$  and  $(\beta, X, \{q\})$  are in contact,  $(\alpha, U, \{p\}) \perp (\beta, X, \{q\})$ , if and only if there exists  $(x_i) \subset \text{dom}(\alpha) \cap \text{dom}(\beta)$  so that  $(\alpha(x_i)) \rightarrow p$  and  $(\beta(x_i)) \rightarrow q$ . The two boundary points  $[(\alpha, U, \{p\})], [(\beta, X, \{q\})] \in \mathcal{B}_{\text{ch}}(M)$  are in contact,  $[(\alpha, U, \{p\})] \perp [(\beta, X, \{q\})]$ , if and only if  $(\alpha, U, \{p\}) \perp (\beta, X, \{q\})$ . It is easy to show that the in contact relation is well defined on  $\mathcal{B}_{\text{ch}}(M)$ . If  $(\alpha, U, \{p\}) \not\perp (\beta, X, \{q\})$  then  $(\alpha, U, \{p\})$  and  $(\beta, X, \{q\})$  are said to be separate, denoted  $(\alpha, U, \{p\}) \parallel (\beta, X, \{q\})$ . Likewise, if  $[(\alpha, U, \{p\})] \not\perp [(\beta, X, \{q\})]$  then  $[(\alpha, U, \{p\})]$  and  $[(\beta, X, \{q\})]$  are separate, denoted  $[(\alpha, U, \{p\})] \parallel [(\beta, X, \{q\})]$ .*

The in contact relation expresses the idea of two boundary points in different extensions representing some similar portion of the boundary of the manifold (as their identifying sets of sequences have non-empty intersection). This similar portion of the boundary is the ‘limit point’ of the sequence  $(x_i)$  used in Definition 2.24.

The compatibility condition is:

**Definition 2.25.** *Two extensions  $(\alpha, U)$  and  $(\beta, X)$  are compatible if for all  $p \in BP(\alpha) \cap U$  and  $q \in BP(\beta) \cap X$ ,  $(\alpha, U, \{p\}) \perp (\beta, X, \{q\}) \Rightarrow (\alpha, U, \{p\}) \equiv (\beta, X, \{q\})$ .*

Thus given two extensions a boundary point in one extension can only correspond to a boundary point in the other extension, not to a collection of boundary points. A reflection of this is the following result.

**Proposition 2.26.** *Let  $Q \subset EX(M)$  be a set of pair-wise compatible extensions. Then  $Q(M)$ , given in Definition 2.15, is Hausdorff.*

*Proof.* It is trivial to show that  $\{(x, y) \in N_Q \times N_Q : q(x) = q(y)\}$  is closed by the definition of  $Q(M)$  and the pair-wise compatibility assumption. Therefore as  $Q(M)$  is the image of  $q$  and as  $q$  is open, Theorem 13.10 of [29] implies that  $Q(M)$  is Hausdorff.  $\square$

Thus the compatibility condition ensures that the boundary points  $\partial_{\iota_Q}(M)$  are a nicely behaved topological space. All that is missing for  $Q(M)$  to be a topological manifold with boundary is a suitable collection of charts and a demonstration of second countability. This is where the assumptions of countability and of ‘sufficiently nice extensions’ are required.

**Proposition 2.27.** *Let  $Q \subset EX(M)$  be a countable set of pair-wise compatible extensions so that for each  $(\alpha, U) \in Q$ ,  $N_{(\alpha, U)}$  is a manifold with boundary. Then  $Q(M)$ , as given in Definition 2.15, is a topological manifold with boundary.*

*Proof.* From above  $Q(M)$  is Hausdorff. It is trivial to show that  $Q$  being a countable set implies that  $N_Q$  is second countable. Since  $q$  is open  $Q(M)$  is second countable. Let

$$A = \{\alpha \circ \iota_Q^{-1} : \alpha \in \mathcal{A}(M)\} \cup \bigcup_{(\alpha, U) \in Q} \{f \circ (q|_{N_{(\alpha, U)}})^{-1} : (\alpha, U) \in Q, f \in \mathcal{A}(N_{(\alpha, U)})\}.$$

The collection of maps in  $A$  gives the needed homeomorphisms to  $\mathbb{R}^{n-1} \times \{x \in \mathbb{R} : x \geq 0\}$ . Therefore  $Q(M)$  is a topological manifold with boundary.  $\square$

The transition functions on the boundary  $\overline{\beta \circ \alpha^{-1}}$  now play the role of genuine coordinate transformations as  $f \circ (q|_{N_{(\alpha, U)}})^{-1} \circ q|_{N_{(\beta, V)}} \circ h^{-1} = f \circ \overline{\beta \circ \alpha^{-1}} \circ h^{-1}$  for  $f \in \mathcal{A}(N_{(\alpha, U)})$  and  $h \in \mathcal{A}(N_{(\beta, V)})$ .

If one is willing to assume, for any pair of extensions  $(\alpha, U), (\beta, V) \in Q$  that the extensions of the transition functions  $\beta \circ \alpha$  to some subset of  $\partial_U \text{ran}(\alpha)$  is  $C^k$  then the following corollary holds.

**Corollary 2.28.** *Let  $M$  be a  $C^l$  manifold. If  $Q \subset EX(M)$  is a set of pair-wise compatible extensions and there exists  $k \in \mathbb{N}$ ,  $0 \leq k \leq l$ , so that;*

1. *for all  $(\alpha, U) \in Q$ ,  $N_{(\alpha, U)}$  is a  $C^k$ , manifold with boundary,*
2. *for all  $(\alpha, U), (\beta, V) \in Q$  the functions  $\overline{\beta \circ \alpha^{-1}}$  are  $C^k$ .*

*then  $Q(M)$  is a  $C^k$  manifold with boundary.*

*Proof.* Let  $A$  be as in the proof of Proposition 2.27. We need to check that the transition functions between elements of  $A$  are  $C^k$ . This will demonstrate that  $A$  generates a  $C^k$  atlas for  $Q(M)$ . We have three cases. **Case 1.** Let  $f \circ q|_{N_{(\alpha, U)}}^{-1}$  and  $g \circ q|_{N_{(\beta, U)}}^{-1}$  be in  $A$ . The transition map, by Proposition 2.17, is  $g \circ \overline{\beta \circ \alpha^{-1}} \circ f$ . This is  $C^k$  by assumption. **Case 2.** Let  $f \circ q|_{N_{(\alpha, U)}}^{-1}$  and  $\beta \circ \iota_Q$  be in  $A$ . There are two possible transition functions,  $f \circ q|_{N_{(\alpha, U)}}^{-1} \circ \iota_Q \circ \beta^{-1}$  and  $\beta \circ \iota_Q^{-1} \circ q|_{N_{(\alpha, U)}} \circ f^{-1}$ . It is easy to show that  $\beta \circ \iota_Q^{-1} \circ q|_{N_{(\alpha, U)}} = \overline{\beta \circ \alpha^{-1}}$  and  $q|_{N_{(\alpha, U)}} \circ \iota_Q \circ \beta^{-1} = \overline{\alpha \circ \beta^{-1}}$ . Thus the transition maps amount to  $f \circ \overline{\alpha \circ \beta^{-1}}$  and  $\overline{\beta \circ \alpha^{-1}} \circ f^{-1}$  which are both  $C^k$ . **Case 3.** Let  $\alpha \circ \iota_Q^{-1}$  and  $\beta \circ \iota_Q^{-1}$  be in  $A$ . In this case the transition functions are  $\beta \circ \alpha^{-1}$  and  $\alpha \circ \beta^{-1}$  which are  $C^l$ . Thus  $Q(M)$  equipped with the atlas generated by  $A$  is a  $C^k$  manifold with boundary.  $\square$

There is a converse to Proposition 2.27 and its corollary.

**Proposition 2.29.** *Let  $\phi : M \rightarrow M_\phi$  be an envelopment. Then, in the notation of Proposition 2.9,  $Q = \{(\alpha \circ \phi, \text{ran}(\alpha)) : \alpha \in \mathcal{A}(M_\phi), \text{dom}(\alpha) \cap \partial\phi(M) \neq \emptyset\}$  is a pair-wise compatible set of extensions and there exists a homeomorphism, in the notation of Definition 2.15,  $f : Q(M) \rightarrow \overline{\phi(M)}$  so that  $f \circ \iota_Q = \phi$ ,  $f(q(N_{(\alpha \circ \phi)})) = \text{dom}(\alpha) \cap \phi(M)$  and  $\overline{\beta \circ \phi \circ \phi^{-1} \circ \alpha^{-1}} = \beta \circ \alpha^{-1}$ .*

*Proof.* The proof is long and depends on results in Appendix A so the proof is given in Appendix B.  $\square$

The equation  $\overline{\beta \circ \phi \circ \phi^{-1} \circ \alpha^{-1}} = \beta \circ \alpha^{-1}$  again shows that the coordinate transforms on the boundary are genuine coordinate transformations.

**Corollary 2.30.** *Let  $\phi : M \rightarrow M_\phi$  be an envelopment and let  $Q(M)$  be as given in Definition 2.15. If  $\overline{\phi(M)}$  is a manifold with boundary then  $Q(M)$  is a manifold with boundary and  $f$  is a diffeomorphism, where  $f$  is given in Proposition 2.29.*

*Proof.* The proof depends on the Proposition 2.29 and so has been moved to Appendix B.  $\square$

Proposition 2.29 has an interesting consequence. Given any envelopment  $\phi : M \rightarrow M_\phi$  the topological boundary  $\partial\phi(M)$  is a set of representatives for the set of boundary points  $\partial_Q(M)$ , where  $Q$  is as given in Proposition 2.29. Thus, via Definition 2.22, any ‘boundary’ of the manifold, that can be presented as the topological boundary of an envelopment, induces a set of boundary points  $\sigma_Q \subset \mathcal{B}_{\text{ch}}(M)$  and therefore the boundary given by the embedding is contained in  $\mathcal{B}_{\text{ch}}(M)$ .

A notable example of a boundary defined in this way is Penrose’s conformal boundary. Thus it is in precisely the manner described above that  $\mathcal{B}_{\text{ch}}(M)$  generalises the conformal boundary. In Section 3 I present the concrete calculation of  $\sigma_Q$  for the conformal boundary of Minkowski spacetime.

Lastly, note that Proposition 2.29 did not require the assumption that  $N_{(\alpha \circ \phi, \text{ran}(\alpha))}$  is a manifold with boundary nor did it require that  $Q$  was countable, in contrast to Proposition 2.27. Thus the results of this section are not sharp and it might be possible to improve on Proposition 2.27. Never-the-less Proposition 2.27 will, in most cases, be sufficient, so the generalisation of this proposition has not been sort.

## 2.5. Classification of boundary points

Given an envelopment,  $\phi : M \rightarrow M_\phi$ , of a manifold and a metric or connection on  $M$ , Scott and Szekeres, [25, Section 4], have presented a classification of the points in  $\partial\phi(M)$  into a hierarchy of physically motivated classes and studied the invariance of these classes under the construction of the Abstract Boundary.

Proposition 2.8 shows that each extension  $(\alpha, U)$  induces an envelopment,  $\Psi(\alpha, U) : M \rightarrow M_{(\alpha, U)}$ , so that  $\text{BP}(\alpha) \cap U = \partial\Psi(\alpha, U)(M)$ . Hence Scott and Szekeres’ classification of the elements of  $\partial\Psi(\alpha, U)(M)$  can be used to classify the elements of  $\text{BP}(\alpha) \cap U$ , as long as there is a metric or connection on  $M$ .

Once the classification has been defined, its invariance under the equivalence relation used to define  $\mathcal{B}_{\text{ch}}(M)$  is then discussed. Due to the similarity of the Abstract Boundary and  $\mathcal{B}_{\text{ch}}(M)$  the invariance of the classification under coordinate transformations on the boundary is exactly analogous to the invariance of the classification proved in Scott and Szekeres' paper [25]. The result is a physical classification of  $\mathcal{B}_{\text{ch}}(M)$ . With this in mind, the rest of this section follows the classification of Abstract Boundary points, [25, Section 4], closely. The discussion below is restricted to the case of a metric on  $M$ , all results, however, can be extended in an analogous manner to the case of a connection on  $M$ .

The classification depends on a choice of curves.

**Definition 2.31** ([25, Definition 4]). *A set  $\mathcal{C}$  of parametrized curves in  $M$  is said to have the bounded parameter property (b.p.p.) if the following conditions are satisfied;*

1. *for all  $p \in M$  there exists  $\gamma \in \mathcal{C}$  so that  $p \in \gamma$ ,*
2. *if  $\gamma \in \mathcal{C}$  and  $\delta$  is a sub-curve of  $\gamma$  then  $\delta \in \mathcal{C}$ ,*
3. *for all  $\gamma, \delta \in \mathcal{C}$ , if  $\delta$  is obtained from  $\gamma$  by a change of parameter then either both curves are bounded or both are unbounded.*

Given  $M$  a manifold and  $(\alpha, U) \in \text{EX}(M)$  the classification begins by dividing the elements of  $\text{BP}(\alpha) \cap U$  into four classes.

**Definition 2.32.** *Let  $(M, g)$  be a  $C^l$  pseudo-Riemannian manifold. The boundary point  $(\alpha, U, \{p\})$  is  $C^k$  regular if there exists an open neighbourhood  $V \subset U$  of  $p$  and a  $C^k$  pseudo-Riemannian metric  $\hat{g}$  on  $V$  so that  $(V, \hat{g})$  is an extension of  $(\text{ran}(\alpha) \cap V, (\alpha^{-1})^*(g|_{\alpha^{-1}(\text{ran}(\alpha) \cap V)}))$  as a manifold with pseudo-Riemannian metric. If  $(\alpha, U, \{p\})$  is not  $C^k$  regular then it is  $C^k$  non-regular. The reference to  $\alpha$  and  $U$  will be dropped if clear from context. When the degree of regularity is unimportant it too will be dropped.*

**Definition 2.33.** *The boundary point  $(\alpha, U, \{p\})$  is approachable if there exists  $\gamma \in \mathcal{C}$  so that  $p$  is in the closure of  $\alpha \circ \gamma|_{\gamma^{-1}(\gamma \cap \text{dom}(\alpha))}$ . The point  $p$  is unapproachable if it is not approachable. The reference to  $\alpha$  and  $U$  will be dropped if clear from context.*

The four classes, mentioned above, are approachable regular points, approachable non-regular points, unapproachable regular points and unapproachable non-regular points. With these classes defined, and with the relation  $\triangleright$ , the classification of boundary points can follow the classification in [25] without change.

By virtue of the choice of curves (which should be taken as large as possible, e.g. the set of all curves with generalized affine parameter [17]) unapproachable points are considered to be physically unimportant, even though unapproachable regular points may exist.

**Example 2.34.** *Continuing from Example 2.10. Let  $g$  be the standard metric on  $\mathbb{R}^2$  then  $h = g|_{TM \times TM}$  is a metric on  $M$  so that  $(\mathbb{R}^2, g)$  is an extension of  $(M, h)$ . Let  $\mathcal{C}$  be the set of all affinely parametrised geodesics with respect to  $h$ . The set  $\mathcal{C}$  satisfies the bounded parameter property. With respect to this choice of curves  $(\alpha, U, \{(0, 1)\})$  is a non-approachable regular boundary point.*

Approachable regular points are points through which the manifold can be extended. Approachable non-regular points are elements of the boundary that are physically relevant and through which the manifold cannot be extended. They can be further subdivided into points at infinity and singular points.

**Definition 2.35.** Let  $\mathcal{C}$  be a set of curves with the b.p.p. A boundary point  $(\alpha, U, \{p\})$  is a  $C^k$  point at infinity if;

1.  $p$  is not a  $C^k$  regular boundary point,
2.  $p$  is approachable,
3. for all  $\gamma \in \mathcal{C}$ , if  $\gamma$  approaches  $p$  then  $\gamma$  is unbounded.

The reference to  $\alpha$  and  $U$  will be dropped if clear from context. When the degree of regularity is unimportant it too will be dropped. Thus a  $C^k$  point at infinity will sometimes be referred to as a point at infinity.

**Definition 2.36.** Let  $\mathcal{C}$  be a set of curves with the b.p.p. A boundary point  $(\alpha, U, \{p\})$  is a  $C^k$  singularity if;

1.  $p$  is not a  $C^k$  regular boundary point,
2. there exists  $\gamma \in \mathcal{C}$  so that  $\gamma$  approaches  $p$  with bounded parameter.

The reference to  $\alpha$  and  $U$  will be dropped if clear from context. When the degree of regularity is unimportant it too will be dropped. Thus a  $C^k$  singularity will sometimes be referred to as a singularity.

Approachable non-regular points may be non-regular due to properties of the chart rather than any inherent property of the manifold.

**Definition 2.37.** A point at infinity  $(\alpha, U, \{p\})$  is removable if there exists a boundary set  $(\beta, X, Y)$  so that  $(\beta, X, Y) \triangleright (\alpha, U, \{p\})$  and for all  $y \in Y$ ,  $(\beta, X, \{y\})$  is a regular boundary point.

A singular point  $(\alpha, U, \{p\})$  is removable if there exists a boundary set  $(\beta, X, Y)$  so that  $(\beta, X, Y) \triangleright (\alpha, U, \{p\})$  and for all  $y \in Y$ ,  $(\beta, X, \{y\})$  is a regular boundary point, a point at infinity or unapproachable.

A non-regular boundary point that is either a removable point at infinity or a removable singularity will be called a removable boundary point. A boundary point that is not removable will be called essential.

The definition of a removable boundary point  $(\alpha, U, \{p\})$ , covered by the boundary set  $(\beta, X, Y)$  (so that for all  $y \in Y$ ,  $(\beta, X, \{y\})$  is a regular boundary point if  $p$  is a point at infinity or  $(\beta, X, \{y\})$  is a regular point, unapproachable point or point at infinity if  $p$  is a singular point) is such that the chart  $\beta$  resolves the non-regular behaviour of  $p$  by representing  $[p] \in \mathcal{B}_{\text{ch}}(M)$  by the set of boundary points  $\{[y] : y \in Y\}$ . Thus  $\beta$  removes the poor behaviour of  $\alpha$ .



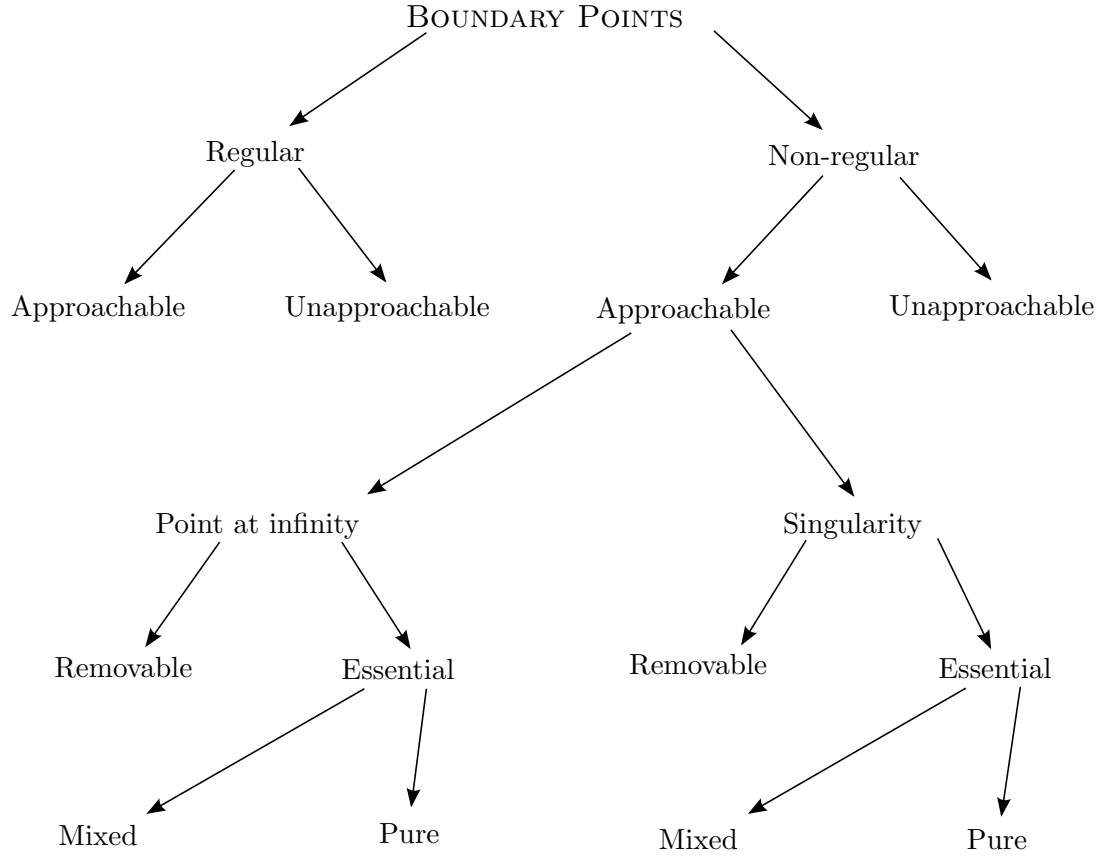


Figure 1: A graphic representation of the classification given in Section 2.5

Approachable non-regular points may cover a regular boundary point of some other extension. Such approachable non-regular points will necessarily exhibit directional behaviour. That is, given a curve whose endpoint is such a point at infinity or singularity, certain physical quantities, e.g. the Kretschmann scalar, could have either regular or singular behaviour.

The Curzon singularity is presented in Section 3.2, see [23, 24]. This is a well known directional singularity. The Curzon singularity covers regular points, points at infinity and unapproachable points once the singularity is expressed in a suitable chart.

**Definition 2.38.** *An essential boundary point  $(\alpha, U, \{p\})$  is mixed, or directional, if there exists a regular boundary point  $(\beta, X, \{q\})$  so that  $(\alpha, U, \{p\}) \triangleright (\beta, X, \{q\})$ . An essential boundary point that is not mixed is called pure.*

Figure 1 gives a summary of the classification.

Extensions that include mixed or removable points are undesirable since this behaviour is considered to be an artefact of the particular chart used. It is a long standing problem in General Relativity to show that all directional singularities can be expressed, as in

the case of the Curzon solution [23, 24], as a collection of genuine singularities, regular points, points at infinity and unapproachable points. In our context this problem can be phrased as: does every manifold have a complete boundary that does not contain removable or mixed boundary points?

As mentioned above invariance under  $\equiv$  is necessary before any quantity defined on the boundary  $\text{BP}(\alpha) \cap U$  of an extension  $(\alpha, U) \in \text{EX}(M)$  can be considered to be coordinate invariant. It turns out that not all the classes of the classification of  $\text{BP}(\alpha) \cap U$  are invariant under  $\equiv$ .

**Proposition 2.39.** *The following classification of  $\mathcal{B}_{ch}(M)$  is well defined. Let  $\mathcal{C}$  be a set of curves with the b.p.p. A boundary point  $[(\alpha, U, \{p\})] \in \mathcal{B}_{ch}(M)$  is;*

1. *approachable if and only if  $(\alpha, U, \{p\})$  is approachable,*
2. *a  $C^k$  indeterminate boundary point if and only if  $(\alpha, U, \{p\})$  is a  $C^k$  regular point, a  $C^k$  removable point at infinity or a  $C^k$  removable singularity,*
3. *is a  $C^k$  point at infinity if and only if  $(\alpha, U, \{p\})$  is an essential  $C^k$  point at infinity,*
4. *is a  $C^k$  singularity if and only if  $(\alpha, U, \{p\})$  is an essential  $C^k$  singularity.*

*Furthermore, an essential boundary point  $[(\alpha, U, \{p\})]$  is mixed, or directional, if and only if  $(\alpha, U, \{p\})$  is mixed. The boundary point  $[(\alpha, U, \{p\})]$  is pure if and only if  $(\alpha, U, \{p\})$  is pure.*

*Proof.* The proof of this is very long as it, essentially, involves checking each case individually. Moreover the full proof is virtually identical to that give in [25], hence I give only a sketch of the result.

Definitions 2.32 and 2.33 correspond, via Propositions 2.8 and 2.9, to the Abstract Boundary definitions of a regular point and an approachable point, ([25, Definitions 28 and 12]), respectively. Moreover, again via Propositions 2.8 and 2.9, the boundary point  $(\alpha, U, \{p\})$  covers the boundary point  $(\beta, V, \{q\})$  if and only if the point  $p \in \partial\Psi(\alpha, U)(M)$  covers  $q \in \partial\Psi(\beta, V)(M)$  according to the Abstract Boundary definition of covers, [25, Definition 14] (see also [25, Theorem 19]). It is now clear that the definitions of the various classes of the classification given here correspond to the definitions given in [25]. Thus the results of Table 1 of [25] are valid for the classifications defined here. Table 1 gives, for each pair of classes, the possibility for the first class to be covered by the second class. Hence, via Table 1, it is possible to determine the invariance of each class under the covering relation. From above it is clear that the same results as for the Abstract Boundary apply to the classification given here, thus the result is proved, see [25, Section 5].  $\square$

Figure 2 gives a summary of the coordinate independent classification.

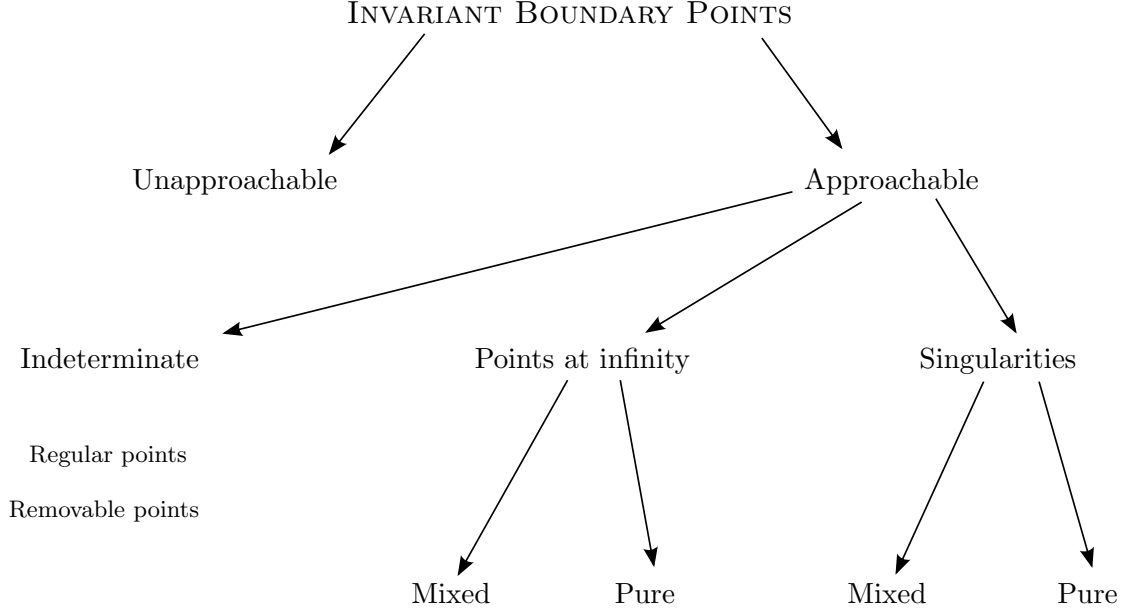


Figure 2: A graphic representation of the coordinate invariant classification given in Section 2.5

### 3. Examples

This section provides three examples of global analysis which I consider typical of the chart based approach. In each case only material from the sources is used to construct and classify a complete representation of the boundary. It is hoped, therefore, that these examples will demonstrate to the reader that Section 2 presents a useful formalization of ‘the chart based approach to studying global structure’.

The first example is the construction of Penrose’s conformal boundary for Minkowski spacetime, as it is given in [18, Section 5.1] (note that this presentation of the boundary differs from Penrose’s own publications [20, 22, 21], in particular it includes spacelike and timelike infinity). The charts used to produce the conformal boundary also induce a boundary, as in Definition 2.22, so that every point of this boundary is a pure point at infinity. This example is presented in great detail in order to provide examples of compatible charts, the construction of a boundary corresponding to collections of compatible charts and how coordinate singularities can be handled.

The second example is the construction of the maximal extension of the Curzon solution given by Scott and Szekeres, [23, 24]. The focus of this example is on how different charts can induce boundaries with different properties. Thus issues arising from coordinate singularities and restrictions of coordinates are ignored. An example of a boundary containing a directional singularity is produced along with a demonstration of how a clever choice of coordinates can ‘resolve’ the directional singularity. The boundary induced by this ‘better’ chart contains regular points, pure points at infinity, pure singularities and unapproachable points.

The third example is the analysis of the global structure of smooth Gowdy symmetric generalized Taub-NUT solutions with spatial  $S^3$ -topology given by Beyer and Hennig, [5]. The spacetimes are presented as the solutions to a system of partial differential equations (PDEs). A closed form for the metric of these space times is not given. Using Beyer and Hennig’s analysis of these equations it is possible to show, however, that the boundary of these spacetimes can contain regular points and essential singularities. From the analysis of [5] it is not possible to conclude that the singularities are pure.

The  $c$ -boundary [9] and the causal boundary [12], require knowledge of the causal structure for their construction. The chart based global analysis of the second and third examples, however, do not determine the causal structure across the whole manifold. In the case of the second example, the majority of the analysis is performed using collections of spacelike curves. In particular the geodesic equations are not solved in a neighbourhood of the singularity and hence additional work would be required before using either the  $c$ -boundary or the causal boundary to analyse the global structure of the singularity. In the case of the third example the properties of the metric are studied on particular curves along which the PDEs reduce to ODEs. Not only is the causal structure on the manifold not discussed, but exact solutions of the PDEs are not given except on these curves. Thus additional work would be required before using either the  $c$ -boundary or the causal boundary to analyse the global structure of solutions to the PDEs. These comments also apply to the  $g$  and  $b$ -boundaries. In this sense  $\mathcal{B}_{\text{ch}}(M)$  is an easier construction to work with.

### 3.1. Minkowski spacetime

This example follows the coordinate construction of Penrose’s conformal boundary for Minkowski spacetime as given in [18, Section 5.1] (see [20, 22, 21] for Penrose’s own presentations, which differ slightly) and shows that the charts used to construct the conformal boundary also admit extensions that give a representation of Penrose’s conformal boundary within  $\mathcal{B}_{\text{ch}}(M)$ , see Section 2.4. The standard analysis of affinely parametrised geodesics in Minkowski spacetime implies that every element of this induced boundary is a pure point at infinity.

Minkowski spacetime is  $M = \mathbb{R}^4$  equipped with the metric  $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$  for  $(t, x, y, z) \in \mathbb{R}^4$ . This chart gives no information about the boundary of the spacetime as its domain is complete, see Definition 2.1. In order to introduce admissible boundary points it is necessary to find charts whose images in  $\mathbb{R}^4$  have non-empty topological boundaries. The construction of  $\mathcal{B}_{\text{ch}}(M)$  provides no information about which charts to use. Indeed  $\mathcal{B}_{\text{ch}}(M)$  is chart agnostic. Two obvious examples are the one-point compactification of  $M$  and an embedding of  $M$  into the 4-dimensional unit ball. Both of these charts would induce complete boundaries, but neither is suitable for the material of this section. Here, I am showing how the constructions of the previous section relate to Penrose’s conformal compactification of Minkowski space. Hence the charts used must behave appropriately with respect to this conformal structure. Therefore the conformal structure of Minkowski space is an additional piece of physical information that is being used to provide guidance on the construction of the charts of this section. In general,

when working with an arbitrary manifold, the selection of appropriate charts will require geometric or physical insight, just as is currently required for the chart based approach to the study of global structure.

Define a new chart by

$$\begin{aligned}\alpha_1(t, x, y, z) &= \left( t, \sqrt{x^2 + y^2 + z^2}, \cos^{-1} \left( \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right), \arctan(x, y) \right) \\ &= (t, r_1, \theta_1, \phi_1),\end{aligned}$$

where  $\arctan : \mathbb{R} \times \mathbb{R} \rightarrow (-\pi, \pi)$  is as given in Example 2.2. Using  $t, x, y, z$  coordinates, let  $O = \{(t, 0, 0, 0) : t \in \mathbb{R}\}$ ,  $X = \{(t, x, 0, z) : t, z \in \mathbb{R}, x < 0\}$  and  $Z = \{(t, 0, 0, z) : t \in \mathbb{R}, z \in \mathbb{R}, z \neq 0\}$ . In order to avoid coordinate singularities, and because of the requirement that both  $\text{dom}(\alpha_1)$  and  $\text{ran}(\alpha_1)$  be open, it is necessary to take  $\text{dom}(\alpha_1) = \mathbb{R}^4 \setminus (O \cup X \cup Z)$  and  $\text{ran}(\alpha_1) = \mathbb{R} \times \mathbb{R}^+ \times (0, \pi) \times (-\pi, \pi)$ . To cover the whole manifold, take a second chart using spherical coordinates centred on  $(0, 1, 1, 0)$  with the  $z$  and  $y$  axis swapped and the  $x$ -axis taken to  $-x$ -axis. Note that these are the spherical coordinates based on a translated and rotated Cartesian frame with respect to the frame given by the  $t, x, y, z$  coordinates. This ensures that the new chart appropriately resolves the points in  $M$  that are not in  $\text{dom}(\alpha_1)$  as  $r_1 \rightarrow \infty$ . The equation for the new chart is

$$\begin{aligned}\alpha_2(t, x, y, z) &= \left( t, \sqrt{(x-1)^2 + (y-1)^2 + z^2}, \right. \\ &\quad \left. \cos^{-1} \left( \frac{y-1}{\sqrt{(x-1)^2 + (y-1)^2 + z^2}} \right), \arctan(1-x, z) \right) \\ &= (t, r_2, \theta_2, \phi_2).\end{aligned}$$

Similarly to  $\alpha_1$ ,

$$\begin{aligned}\text{dom}(\alpha_2) &= \mathbb{R}^4 \setminus \left( \{(t, 1, 1, 0) : t \in \mathbb{R}\} \right. \\ &\quad \left. \cup \{(t, x, y, 0) : t, y \in \mathbb{R}, x > 1\} \cup \{(t, 1, y, 0) : t \in \mathbb{R}, y \in \mathbb{R}, y \neq 0\} \right)\end{aligned}$$

and  $\text{ran}(\alpha_2) = \mathbb{R} \times \mathbb{R}^+ \times (0, \pi) \times (-\pi, \pi)$ .

The topological boundary,  $\partial \text{ran}(\alpha_i)$ , of  $\text{ran}(\alpha_i)$ ,  $i \in \{1, 2\}$ , in  $\mathbb{R}^4$  is the union of the disjoint sets

$$\mathbb{R} \times \{0\} \times [0, \pi] \times [-\pi, \pi] \tag{1}$$

$$\mathbb{R} \times \mathbb{R}^+ \times [0, \pi] \times \{-\pi, \pi\} \tag{2}$$

$$\mathbb{R} \times \mathbb{R}^+ \times \{0, \pi\} \times [-\pi, \pi]. \tag{3}$$

A sequence  $(x_k) \subset \text{dom}(\alpha_i)$ ,  $i \in \{1, 2\}$ , so that  $(\alpha_i(x_k))$  converges to;

- a point in set (1) will converge to a point in  $O$ , if  $i = 1$ , or, where  $(t, x, y, z)$  coordinates are used,  $\{(t, 1, 1, 0) : t \in \mathbb{R}\}$ , if  $i = 2$ ,
- a point in set (2) will converge to a point in  $X$ , if  $i = 1$ , or, where  $(t, x, y, z)$  coordinates are used,  $\{(t, x, y, 0) : t, y \in \mathbb{R}, x > 1\}$ , if  $i = 2$ ,
- a point in set (3) will converge to a point in  $Z$ , if  $i = 1$ , or, where  $(t, x, y, z)$  coordinates are used,  $\{(t, 1, y, 0) : t \in \mathbb{R}, y \in \mathbb{R}, y \neq 0\}$ , if  $i = 2$ .

Thus no point in the union of the sets (1), (2) and (3) is an admissible boundary point.

Some form of compactification of the coordinates will introduce admissible boundary points. As mentioned before the guiding principle in this section is the conformal structure of  $M$ . Since  $t - r$  and  $t + r$  are coordinates with null tangent vectors they have a form of invariance with respect to conformal transformations of  $M$ . Hence rather than compactifying  $t$  and  $r$ , a change of coordinates to  $t + r$  and  $t - r$  will be performed and then compactification of these new coordinates will be done. The result is below. Before I present this, it is worth pointing out one particular consequence of this choice. Because null geodesics in Minkowski space are necessarily traveling forward or backward in time, “infinity” for each fixed  $t$  slice of  $M$  will be identified on the boundary. Below, this identified point is denoted  $\iota_i^0$ . There are charts that respect the conformal structure of  $M$  and which do not perform this identification, see for example [11, 4], but they shall not be needed here. As I have mentioned before, there is great freedom in the choice of what chart to use to build a local representation of the boundary of  $M$ . Because of this additional information, e.g. geometric or physical, something beyond the purely topological is needed to inform any choice.

Let  $i \in \{1, 2\}$  and define

$$\beta_i(t, r_i, \theta_i, \phi_i) = (\tan^{-1}(t + r_i), \tan^{-1}(t - r_i), \theta_i, \phi_i) = (p_i, q_i, \theta_i, \phi_i).$$

Hence  $\text{dom}(\beta_i) = \text{dom}(\alpha_i)$  and

$$\text{ran}(\beta_i) = \left\{ (p_i, q_i) : p_i, q_i \in \left( -\frac{1}{2}\pi, \frac{1}{2}\pi \right), q_i \leq p_i \right\} \times (0, \pi) \times (-\pi, \pi).$$

The boundary,  $\partial \text{ran}(\beta_i)$ , of  $\text{ran}(\beta_i)$ ,  $i \in \{1, 2\}$ , in  $\mathbb{R}^4$  is given by the union of the sets

$$\left\{ (p_i, q_i) : p_i, q_i \in \left( -\frac{1}{2}\pi, \frac{1}{2}\pi \right), q_i \leq p_i \right\} \times \{0, \pi\} \times [-\pi, \pi], \quad (4)$$

$$\left\{ (p_i, q_i) : p_i, q_i \in \left( -\frac{1}{2}\pi, \frac{1}{2}\pi \right), q_i \leq p_i \right\} \times [0, \pi] \times \{-\pi, \pi\}, \quad (5)$$

and the sets

$$\begin{aligned}
\mathcal{J}_i^+ &= \left\{ \frac{1}{2}\pi \right\} \times \left( -\frac{1}{2}\pi, \frac{1}{2}\pi \right) \times (0, \pi) \times (0, 2\pi), \\
\mathcal{J}_i^- &= \left( -\frac{1}{2}\pi, \frac{1}{2}\pi \right) \times \left\{ \frac{1}{2}\pi \right\} \times (0, \pi) \times (0, 2\pi), \\
\mathfrak{I}_i^+ &= \left\{ \frac{1}{2}\pi \right\} \times \left\{ \frac{1}{2}\pi \right\} \times (0, \pi) \times (0, 2\pi), \\
\mathfrak{I}_i^0 &= \left\{ \frac{1}{2}\pi \right\} \times \left\{ -\frac{1}{2}\pi \right\} \times (0, \pi) \times (0, 2\pi), \\
\mathfrak{I}_i^- &= \left\{ -\frac{1}{2}\pi \right\} \times \left\{ -\frac{1}{2}\pi \right\} \times (0, \pi) \times (0, 2\pi).
\end{aligned}$$

As before the elements of the sets (4) and (5) are not admissible boundary points. The other sets correspond to the usual expression of the conformal boundary of Minkowski space with respect to the chart  $\beta_i$ . The sets have been labelled so that this correspondence is obvious, e.g. future null infinity is represented by  $\mathcal{J}_1^+$  and  $\mathcal{J}_2^+$  for  $\beta_1$  and  $\beta_2$  respectively, see [18, page 123].

Since the ranges of  $\beta_1$  and  $\beta_2$  are the same, they have the same admissible boundary points, e.g.  $\mathcal{J}_1^+ = \mathcal{J}_2^+$ . It is important, however, to maintain a distinction between these admissible boundary points since the domains of  $\beta_1$  and  $\beta_2$  are different and therefore their admissible boundary points represent different parts of the boundary of  $M$ . Letting  $W = \mathbb{R} \times \mathbb{R} \times (0, \pi) \times (-\pi, \pi)$  the extension  $(\beta_i, W) \in \text{EX}(M)$ , for  $i \in \{1, 2\}$ , is such that  $\partial_W \text{ran}(\beta_i)$  is the union of the sets  $\mathcal{J}_i^+, \mathcal{J}_i^-, \mathfrak{I}_i^+, \mathfrak{I}_i^0$  and  $\mathfrak{I}_i^-$ . Thus each extension,  $(\beta_i, W)$ , induces a boundary,  $\sigma_i$ , where  $[(\beta_i, W, \{p\})] \in \sigma_i$  if and only if  $p$  is an element of one of  $\mathcal{J}_i^+, \mathcal{J}_i^-, \mathfrak{I}_i^+, \mathfrak{I}_i^0$  and  $\mathfrak{I}_i^-$ .

The two sets  $\sigma_1$  and  $\sigma_2$  are not equal. For example, since  $X$  is not in the domain of  $\beta_1$  any sequence lying in  $X$  will not converge to a boundary point in  $\sigma_1$ . However,  $X$  is in the domain of  $\beta_2$  thus there are boundary points representing the ‘limit points of  $X$  at infinity’ in  $\sigma_2$ . Since

$$\begin{aligned}
\beta_2(X) &= \left\{ \left( \tan^{-1} \left( t + \sqrt{(x-1)^2 + 1 + z^2} \right), \right. \right. \\
&\quad \left. \tan^{-1} \left( t - \sqrt{(x-1)^2 + 1 + z^2} \right), \right. \\
&\quad \left. \cos^{-1} \left( \frac{-1}{\sqrt{(x-1)^2 + 1 + z^2}} \right), \arctan(1-x, z) \right) : t, z \in \mathbb{R}, x < 0 \right\},
\end{aligned}$$

for  $z = 0$  and any fixed  $t$ , the limit point given by  $x \rightarrow -\infty$  is represented by the boundary point  $(\beta_2, W, (\frac{\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, 0))$ . Thus

$$\left[ \left( \beta_2, W, \left( \frac{\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, 0 \right) \right) \right] \in \sigma_2.$$

Suppose that exists  $p = [(\beta_1, W, (p', q', \theta, \phi))] \in \sigma_1$  so that

$$p = \left[ \left( \beta_2, W, \left( \frac{\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, 0 \right) \right) \right].$$



Then, by definition, it must be the case that every sequence  $(x_j) \subset \text{dom}(\beta_2)$  so that  $\beta_2(x_j) \rightarrow (\frac{\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, 0)$  is such that there exists a subsequence  $(y_j) \subset (x_j)$  so that  $(y_j) \subset \text{dom}(\beta_1)$  and  $\beta_1(y_j) \rightarrow (p', q', \theta, \phi)$ . From above there exists a sequence  $(x_i) \subset X$  so that  $\beta_2(x_i) \rightarrow (\frac{\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, 0)$ . By construction, however,  $X \not\subset \text{dom}(\beta_1)$  and therefore there is no suitable subsequence of  $(x_i)$ . Hence, for all  $p \in \sigma_1$ ,  $p \neq [(\beta_2, W, (\frac{\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, 0))]$ . Therefore  $\sigma_1 \neq \sigma_2$ .

Since  $M = \text{dom}(\beta_1) \cup \text{dom}(\beta_2)$  and both  $\text{ran}(\beta_1)$  and  $\text{ran}(\beta_2)$  are compact, any sequence in  $M$  that does not converge to a point in  $\sigma_i$  will converge to a point in  $\sigma_j$  where  $i, j \in \{1, 2\}$  with  $i \neq j$ . Thus the union  $\sigma = \sigma_1 \cup \sigma_2$  represents all of ‘the boundary’ of  $M$ , as can be expected due to the relationship with the conformal boundary. Hence  $\sigma$  is complete.

The extensions  $(\beta_1, W)$  and  $(\beta_2, W)$  are compatible. Let  $p = [(\beta_1, W, \{x\})]$  and  $q = [(\beta_2, W, \{y\})]$ . Assume that  $p \perp q$ , hence there exists  $(z_i) \subset M$  so that  $\beta_1(z_i) \rightarrow x$  and  $\beta_2(z_i) \rightarrow y$ . Let  $(x_i) \subset M$  be such that  $\beta_1(x_i) \rightarrow x$ . For each  $i \in \mathbb{N}$ , let  $\hat{x}_{2i} = x_i$  and  $\hat{x}_{2i+1} = z_i$ . By construction  $\beta_1(\hat{x}_i) \rightarrow x$ . The transition function,  $\beta_2 \circ \beta_1^{-1}$ , amounts to the translation and rotation used to transform between the two Cartesian frames used to generate the spherical polar coordinates  $\alpha_1$  and  $\alpha_2$ . Therefore the transition function  $\beta_2 \circ \beta_1^{-1}$  preserves distances. This implies that  $(\beta_2(\hat{x}_i))$  is Cauchy with respect to the Euclidean distance on  $\text{ran}(\beta_2)$ . Since  $(\beta_2(\hat{x}_i))$  is Cauchy,  $(z_i) \subset (\hat{x}_i)$  and as  $\beta_2(z_i) \rightarrow y$  it is the case that  $\beta_2(\hat{x}_i) \rightarrow y$ . As  $(x_i) \subset (\hat{x}_i)$  the sequence  $(\beta_2(x_i))$  converges to  $y$ . Therefore  $q \triangleright p$ . Repeating the argument with the roles of  $x$  and  $y$  interchanged demonstrates that  $q = p$  as required. Hence  $(\beta_1, W)$  and  $(\beta_2, W)$  are compatible.

I compute a specific example. Consider the line given by  $L = \{(0, x, -x, 1) : x > 0\}$ . Thus

$$\beta_1(L) = \left\{ \left( \tan^{-1}(\sqrt{2x^2 + 1}), \tan^{-1}(-\sqrt{2x^2 + 1}), \cos^{-1}\left(\frac{1}{\sqrt{2x^2 + 1}}\right), -\frac{\pi}{4} \right) : x > 0 \right\}$$

and

$$\beta_2(L) = \left\{ \left( \tan^{-1}(\sqrt{2x^2 + 3}), \tan^{-1}(-\sqrt{2x^2 + 3}), \cos^{-1}\left(\frac{-x-1}{\sqrt{2x^2 + 3}}\right), \arctan(1-x, 1) \right) : x > 0 \right\}.$$

Hence, the boundary points representing the ‘end-point’ of the line at infinity are  $[(\beta_1, W, (\frac{\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{4}))] \in \sigma_1$  and  $[(\beta_2, W, (\frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{4}, \pi))] \in \sigma_2$ . Choosing a sequence  $(x_i) \subset L$  with no limit point,  $\beta_1(x_i) \rightarrow (\frac{\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{4})$  and  $\beta_2(x_i) \rightarrow (\frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{4}, \pi)$ . The paragraph above implies that

$$\left[ \left( \beta_1, W, \left( \frac{\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{4} \right) \right) \right] = \left[ \left( \beta_2, W, \left( \frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{4}, \pi \right) \right) \right].$$

Lastly, note that the points in  $\sigma_2 \setminus \sigma_1$  are exactly those points which represent the endpoints of  $X$  and  $Z$ . These are the points lying on the ‘sphere at infinity’ which cannot be expressed in the chart  $\beta_1$ .

I give an example of how these points can be calculated. Let  $\gamma : (0, \infty) \rightarrow M$  be given, in  $t, x, y, z$  coordinates, by  $\gamma(\tau) = (\sqrt{(\tau+1)^2 + 1 + \tau^2}, -\tau, 0, \tau)$ . This curve lies in  $X$  for  $\tau < 0$ . From the calculation of  $\beta_2(X)$ ,

$$\beta_2 \circ \gamma(\tau) = \left( \tan^{-1} \left( 2\sqrt{(\tau+1)^2 + 1 + \tau^2} \right), 0, \right. \\ \left. \cos^{-1} \left( \frac{-1}{\sqrt{(\tau+1)^2 + 1 + \tau^2}} \right), \arctan(1 + \tau, \tau) \right)$$

so that as  $\tau \rightarrow \infty$  the curve  $\beta_2 \circ \gamma(\tau)$  limits to the boundary point

$$\left[ \left( \beta_2, X, \left( \frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{4} \right) \right) \right].$$

The point  $(\frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{4})$  is an element of  $\mathcal{J}_2^+$ . Since  $\gamma \subset X$  and  $X \not\subset \text{dom}(\beta_1)$  this boundary point is an element of future null infinity that cannot be represented by the chart  $\beta_1$ .

Taking the set of all affinely parametrised geodesics as the b.p.p. satisfying set of curves, the usual analysis of the Penrose conformal boundary implies that every point in  $\sigma$  is a pure point at infinity.

### 3.2. Curzon solution

In this section I present the construction and classification of two representations of the boundary of the Curzon solution. The Curzon solution is one of the better known and analysed examples of a directional singularity, [23, Section 1]. I do not go into as much detail as Section 3.1 and assume that, with the results of Section 3.1 as a guide, the reader can fill in the details.

The Curzon solution is the Weyl metric, [26], for a monopole potential, [8]. The manifold is  $M = \mathbb{R} \times (\mathbb{R}^2 \setminus \{(0, 0)\}) \times \mathbb{S}^1$ . Letting  $\phi^{-1} : (0, 2\pi) \rightarrow \mathbb{S}^1$  be defined by  $\phi^{-1}(\theta) = (\cos \theta, \sin \theta)$  there is a chart,  $\alpha$ , given by  $\alpha(t, z, r, s) = (t, z, r, \phi(s))$ . An additional chart is required to cover all of  $M$  to account for the coordinate singularity introduced by  $\phi$ . In the chart  $\alpha$  the metric takes the form,  $ds^2 = -\exp(2\lambda)dt^2 + \exp(2\nu - 2\lambda)(dr^2 + dz^2) + r^2 \exp(-2\lambda)d\phi^2$ , where  $\lambda = \frac{-m}{R}$ ,  $\nu = \frac{-m^2 r^2}{2R^4}$ ,  $R = \sqrt{r^2 + z^2}$  and  $m \neq 0$ , [23, Equations 1 and 2]. Note that in [24, Equation 1] the metric has been rescaled with respect to  $m$  so that  $\lambda = \frac{-1}{R}$ ,  $\nu = \frac{-r^2}{2m^2 R^4}$  and  $R = \frac{\sqrt{r^2 + z^2}}{m}$ . This rescaling does not effect the coordinate transformations used below nor the analysis of the global structure of the spacetime (compare the metrics and transformations used in [23] and [24]).

For the sake of concentrating on the structure of the boundary at  $R = 0$  I assume that  $ds^2$  approaches the Minkowski metric for  $R$  very large. As a consequence charts similar to those used in the previous section, Section 3.1, can be constructed to induce a boundary

$\sigma \subset \mathcal{B}_{\text{ch}}(M)$  which is comprised entirely of pure points at infinity. This boundary will be such that if  $(x_i)_i \subset M$  is a sequence of points so that if  $R(x_i) \rightarrow \infty$  then there exists at least one element of  $\sigma$  that is approached by  $(x_i)$ , whereas, if  $R(x_i) \rightarrow 0$  then there is no element of  $\sigma$  that is approached by  $(x_i)$ . Hence  $\sigma$  is not complete.

To find a complete boundary, boundary points that represent the ‘points at  $R = 0$ ’ need to be constructed. From the construction of  $\alpha$ ,  $\text{BP}(\alpha) = \{(t, 0, 0, \phi) : t \in \mathbb{R}, \phi \in (0, 2\pi)\}$ , which are exactly the points for which  $R = 0$ . Thus by taking  $U = \mathbb{R}^3 \times (0, 2\pi)$  the set of boundary points,  $\sigma_\alpha = \{[(\alpha, U, \{(t, 0, 0, \phi)\})] : t \in \mathbb{R}, \phi \in (0, 2\pi)\}$  results. This gives a representation of the surface  $R = 0$ . By construction, if  $p \in \sigma_\alpha$  and  $q \in \sigma$  then  $p \parallel q$ . Thus the extensions representing the set of boundary points for  $r \rightarrow \pm\infty$ , mentioned above, and  $(\alpha, U)$  are compatible. From the definition of  $M$ , any sequence in  $M$  that is without limit points must approach at least one boundary point in  $\sigma \cup \sigma_\alpha$  and therefore  $\sigma \cup \sigma_\alpha$  is a complete boundary.

Gautreau and Anderson, [15], have shown that the Kretschmann scalar,  $K$ , for the Curzon solution limits to 0 for curves on the  $z$ -axis as  $R \rightarrow 0$ . But that it limits to  $\infty$  for other straight line directions of approach to  $R = 0$ . Cooperstock and Junevicius, [7], took this analysis further by considering curves defined by  $\frac{z}{m} = C \left(\frac{r}{m}\right)^n$  for  $C, n > 0$ . They found that  $\lim_{R \rightarrow 0} K \rightarrow 0$  for  $0 < n < \frac{2}{3}$  and that  $\lim_{R \rightarrow 0} K \rightarrow \infty$  for  $n > \frac{2}{3}$ . Scott and Szekeres, [23, Section 3], corrected a mistake that Cooperstock and Junevicius made for the critical,  $n = \frac{2}{3}$ , case and showed that, when  $n = \frac{2}{3}$ , it was possible to construct curves so that  $\lim_{R \rightarrow 0} K$  takes any value in  $\mathbb{R}^+ \cup \{0, \infty\}$ . This implies that the elements of  $\sigma_\alpha$  are not regular points.

In order to take the classification of the points in  $\sigma \cup \sigma_\alpha$  further it is necessary to chose a b.p.p. satisfying set of curves. Scott and Szekeres’ analysis of the global structure of the Curzon solution, given in [24], is based on on timelike, null and spacelike geodesics. Hence I take these curves, with affine parameters, as the set of b.p.p. satisfying curves. As seen above, however, the literature prior to [24] used a variety of non-geodesic curves. While these curves cannot contribute to the ‘approachability’ of boundary points (because of the choice of the b.p.p. satisfying set of curves) the analysis of scalar quantities along them will still give information about the regularity of the limit points of their images under the chart  $\alpha$ .

Scott and Szekeres, [23], show that there exist spacelike geodesics that approach the  $R = 0$  surface with bounded affine parameter, [23, Section 3.b]. Thus the elements of  $\sigma_\alpha$  are singular points. It would be nice to conclude from the analysis of the Kretschmann scalar that the elements of  $\sigma_\alpha$  are directional singularities. While the results of the analysis are suggestive of this, they do not prove it. In order to show that  $[(\alpha, U, \{p\})] \in \sigma_\alpha$  is a directional singularity it is necessary to show two things. First, that  $(\alpha, U, \{p\})$  is not covered by non-singular points. Second, a regular boundary point  $(\beta, X, \{q\})$  so that  $(\alpha, U, \{p\}) \triangleright (\beta, X, \{q\})$  needs to be constructed. The analysis of the Kretschmann scalar proves the first but not the second condition. Fortunately, Scott and Szekeres have constructed a chart (presented as  $\beta$  below) which produces a regular point covered by every element of  $\sigma_\alpha$ . Thus the elements of  $\sigma_\alpha$  are indeed directional singularities.

For the purposes of studying the directional behaviour Scott and Szekeres restrict to the half plane  $z \geq 0$  (as the metric has a symmetry under  $z \mapsto -z$ ) and take two

coordinate transformations. The first is, [23, Equations 17 and 18],

$$x(r, z) = \tan^{-1} \left( \frac{r}{m} \exp \left( \frac{m}{z} \right) \right) + \tan^{-1} \left( \frac{r}{m} \exp \left( - \left( \frac{\sqrt{2}m}{r} \right)^{\frac{2}{3}} \right) \right),$$

$$y(r, z) = \tan^{-1} \left( 3 \frac{z}{m} - \left( \frac{z}{m} \right)^2 \frac{R \exp(\nu - \lambda)}{\left( R^{12} + R^4 + \frac{1}{3} \left( \frac{r}{m} \right)^2 \right)^{\frac{1}{4}}} \right),$$

where  $(x, y) \in (-\frac{1}{2}\pi, \frac{1}{2}\pi) \times (-\frac{1}{2}\pi, 0] \cup (-\pi, \pi) \times (0, \frac{\pi}{2})$ . These coordinates were constructed using a combination of numerical calculation of geodesics, physical intuition and trial and error [23, page 566]. The second transformation is, [24, Equations 7 and 8],

$$Y(t, r, z) = \frac{\pi}{2} + \tan^{-1} \left( ay^3 \left( x^2 - \frac{\pi^2}{4} \right)^2 \frac{m}{z} + 3 \frac{z}{m} - \right.$$

$$\left. \left( \frac{z}{m} \right)^2 \frac{R \exp(v - \lambda)}{\left( R^{12} \left( 1 + \left( \frac{t}{m} \right)^4 \right) + R^4 + \frac{1}{3} \left( \frac{r}{m} \right)^2 \right)^{\frac{1}{4}}} \right),$$

$$T(t, r, z) = \tan^{-1} \left( \exp(-\hat{K}) \left( \frac{t}{m} + H \right) + \frac{t}{m} \left( y + \frac{\pi}{2} \right)^3 \right) +$$

$$\tan^{-1} \left( \exp(-\hat{K}) \left( \frac{t}{m} - H \right) + \frac{t}{m} \left( y + \frac{\pi}{2} \right)^3 \right),$$

where

$$H(r, z) = \frac{1}{2} \left( \frac{r}{z} \right)^2 \exp \left( \frac{2m}{z} \right) + \int_1^{\frac{z}{m}} \exp \left( \frac{2}{u} \right) du$$

and

$$\hat{K}(r, z) = \left( y + \frac{\pi}{2} \right) R + \left( \frac{\tan x}{\tan y} \right)^2.$$

Note that in [23, 24] the function  $\hat{K}$  is denoted as  $K$ . I use  $\hat{K}$  here to distinguish it from the Kretschmann scalar. Please refer to [24, Section 3] for a discussion of the motivation behind these coordinates.

The coordinate ranges are  $T \in (-\pi, \pi)$  and  $(x, Y) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times (0, \frac{\pi}{2}] \cup (-\pi, \pi) \times (\frac{\pi}{2}, \pi)$ . This defines a chart

$$\beta(t, z, r, s) = (T(t, r, z), x(r, z), Y(t, r, z), \phi(s))$$

so that  $\text{dom}(\beta) = \text{dom}(\alpha)$  and

$$\text{ran}(\beta) = (-\pi, \pi) \times \left( \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \times \left( 0, \frac{\pi}{2} \right] \cup (-\pi, \pi) \times \left( \frac{\pi}{2}, \pi \right) \right) \times (0, 2\pi). \quad (6)$$

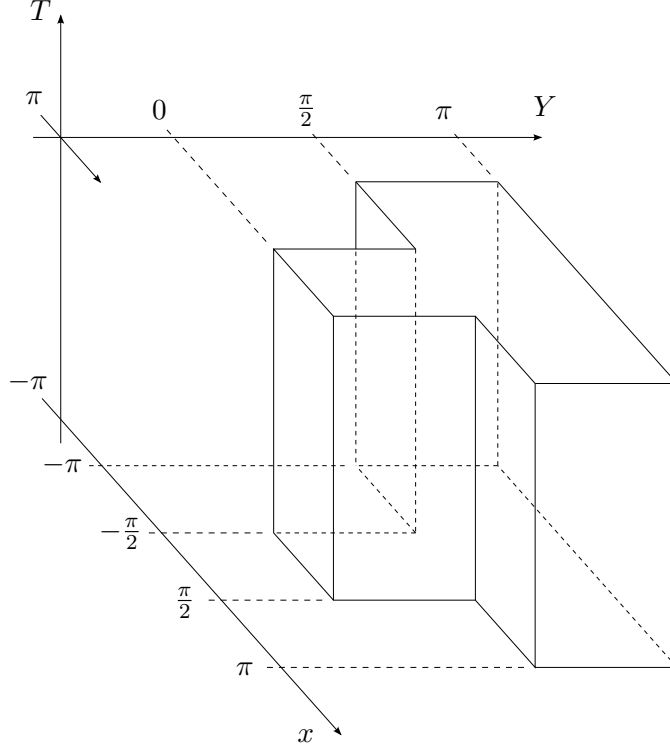


Figure 3: The range of  $\beta$  (Equation (6)), with the  $\phi$  coordinate suppressed, is the interior of the T shaped set.

Figure 3 shows the range of  $\beta$  with the  $\phi$  coordinate suppressed.

The construction of  $\beta$  is based on Scott and Szekeres' detailed analysis of the behaviour the Kretschmann scalar along certain sets of curves, [23, 24]. Hence their construction of this chart is an example of the technique used to study the global structure of a manifold mentioned in the introduction to this paper.

I will now discuss the set of boundary points induced by an extension of  $\beta$ . Because it reduces the complexity of this discussion I will consider  $\beta$  as a chart on the  $z > 0$  submanifold rather than as a chart on  $M$  itself. The topological boundary of the range of  $\beta$  can be divided into the eleven surfaces

$$\begin{aligned}
S_1 &= \left\{ (T, x, \pi, \phi) : T, x \in (-\pi, \pi), \phi \in (0, 2\pi) \right\}, \\
S_2^+ &= \left\{ (T, \pi, Y, \phi) : T \in (-\pi, \pi), Y \in \left(\frac{\pi}{2}, \pi\right), \phi \in (0, 2\pi) \right\}, \\
S_2^- &= \left\{ (T, -\pi, Y, \phi) : T \in (-\pi, \pi), Y \in \left(\frac{\pi}{2}, \pi\right), \phi \in (0, 2\pi) \right\}, \\
S_3^+ &= \left\{ \left(T, x, \frac{\pi}{2}, \phi\right) : T \in (-\pi, \pi), x \in \left(\frac{\pi}{2}, \pi\right), \phi \in (0, 2\pi) \right\}, \\
S_3^- &= \left\{ \left(T, x, \frac{\pi}{2}, \phi\right) : T \in (-\pi, \pi), x \in \left(-\pi, -\frac{\pi}{2}\right), \phi \in (0, 2\pi) \right\},
\end{aligned}$$

$$\begin{aligned}
S_4^+ &= \left\{ \left( T, \frac{\pi}{2}, Y, \phi \right) : T \in (-\pi, \pi), Y \in \left( 0, \frac{\pi}{2} \right), \phi \in (0, 2\pi) \right\}, \\
S_4^- &= \left\{ \left( T, -\frac{\pi}{2}, Y, \phi \right) : T \in (-\pi, \pi), Y \in \left( 0, \frac{\pi}{2} \right), \phi \in (0, 2\pi) \right\}, \\
S_5^- &= \left\{ (T, x, 0, \phi) : T \in (-\pi, 0), x \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right), \phi \in (0, 2\pi) \right\}, \\
S_5^+ &= \left\{ (T, x, 0, \phi) : T \in (0, \pi), x \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right), \phi \in (0, 2\pi) \right\}, \\
S_6 &= \left\{ (\pi, x, Y, \phi) : \phi \in (0, 2\pi), \right. \\
&\quad \left. (x, Y) \in (-\pi, \pi) \times \left( \frac{\pi}{2}, \pi \right) \cup \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \times \left( 0, \frac{\pi}{2} \right] \right\}, \\
S_7 &= \left\{ (-\pi, x, Y, \phi) : \phi \in (0, 2\pi), \right. \\
&\quad \left. (x, Y) \in (-\pi, \pi) \times \left( \frac{\pi}{2}, \pi \right) \cup \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \times \left( 0, \frac{\pi}{2} \right] \right\},
\end{aligned}$$

and the lines or points given by the intersections of the closures of the surfaces. See Figure 4 for a graphical representation. All of these surfaces and the points or lines given by intersections of the closures of the surface, consist of admissible boundary points. Let  $X = \mathbb{R}^4$  then  $(\beta, X)$  is an extension. Considering  $\beta$  as a chart in  $M$ , rather than on the submanifold given by  $z > 0$ , the elements of  $S_3^\pm$  are no longer admissible. See [23, Section 5] or the third and second to last sentences on page 578 of [24].

This second chart,  $\beta$ , also induces a boundary,  $\sigma_\beta$ , as the set of boundary points  $[(\beta, X, \{p\})]$  where  $p$  is an element of one of the eleven surfaces or an element of one of the intersections of the closures of the surfaces given above.

Scott and Szekeres do not explicitly state all the information needed for a complete classification of the boundary points of  $\sigma_\beta$  in [23, 24]. The following list contains those statements that can be derived from comments in their papers. Note that the analysis of the two papers is performed in different coordinate systems,  $x, y, t, \phi$  versus  $x, Y, T, \phi$ . This difference, however, does not effect the conclusions drawn below (for justification refer to the second sentence on page 579 of [24]);

1. the set  $S_1 \cup S_2^+ \cup S_2^-$  and the lines  $\overline{S_1} \cap \overline{S_2^\pm} \setminus \overline{S_6} \cup \overline{S_7}$  correspond to spacelike infinity for the  $(t, r, z, \phi)$  coordinates, i.e. as  $R \rightarrow \infty$ . That is, these surfaces are approached by spacelike geodesics and not approached by non-spacelike geodesics. See the sentence beginning ten lines from the top of page 575 of [24],
2. the surfaces  $S_3^\pm$  consist of  $C^\infty$  regular points. See [23, Section 5] or the third and second to last sentences on page 578 of [24].
3. the lines given by  $\overline{S_3^\pm} \cap \overline{S_4^\pm}$  are the singularity of the Curzon solution. The Kretschmann limits to  $\infty$  along every curve that limits to points in  $\overline{S_3^\pm} \cap \overline{S_4^\pm}$ . This is stated in several places in [24] the clearest is the last sentence of page 571,

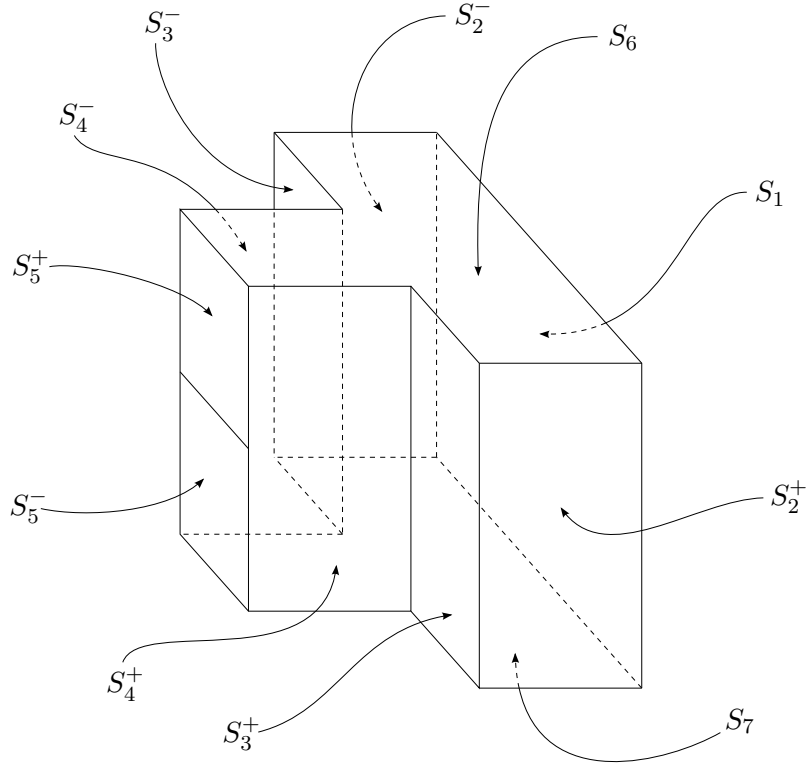


Figure 4: A graphical representation of the eleven surfaces that make up the topological boundary of the range of  $\beta$ , see Equation (6).



4. the set  $S_4^+ \cup S_4^-$  is not approached by any spacelike geodesic. See item 3 on page 568 of [23] and the first paragraph of [24, Section 1],
5. the lines  $(\overline{S_4^\pm} \cap \overline{S_5^+ \cup S_5^-}) \setminus \overline{S_6 \cup S_7}$  and  $\overline{S_5^+} \cap \overline{S_5^-}$  are spacelike infinities, i.e. they are approach by spacelike geodesics and not approached by non-spacelike geodesics. See items 2 and 4 on page 568 of [23],
6. the sets  $S_5^+, S_5^-$  are comprised of  $C^\infty$  regular points. On these surfaces copies of Minkowski spacetime can be smoothly attached. See the last paragraph on page 579 of [24].

The following statements are inferred from [23, 24] but are not explicitly mentioned in either of the papers. Since  $S_3^\pm$  consists of regular boundary points the elements of  $\mathcal{B}_{\text{ch}}(M)$  represented by points in  $(\overline{S_3^\pm} \cap \overline{S_2^\pm}) \setminus \overline{S_6 \cup S_7}$  are implied to be in  $\sigma$ . Nothing is said about future/past timelike/null infinity, though Figure 1 of [24] implies that these will correspond to  $S_6$  and  $S_7$  and the edges  $\overline{S_6} \setminus S_6$  and  $\overline{S_7} \setminus S_7$ . Lastly, nothing is explicitly said about the existence of non-spacelike geodesics that approach the  $S_4^\pm$  surfaces. In principle this information can be determined from [24, Section 2] and the equations for the coordinates given above. It is, however, implied that no non-spacelike geodesic approaches the  $S_4^\pm$  surfaces in the discussion of [24, Section 2] beginning in the last paragraph of page 573. Thus I shall assume that the elements of  $S_4^\pm$  are non-approachable. These claims could be investigated thoroughly. This is beyond the scope of this paper, however, so I will assume that the analysis, as described above, is accurate.

With this in mind then,  $[(\beta, X, \{p\})] \in \sigma_\beta$ , is classified as;

1. an unapproachable point if  $p \in S_4$ ,
2. a  $C^\infty$  indeterminate point if  $p \in S_5^\pm$ ,
3. a pure point at infinity if  $p \in \overline{S_1}, \overline{S_2^\pm}, \overline{S_3^\pm} \cap \overline{S_2^\pm}, \overline{S_6}, \overline{S_7}, \overline{S_5^+} \cap \overline{S_5^-}, \overline{S_4^\pm} \cap (\overline{S_5^+ \cup S_5^-})$ ,
4. a pure singularity if  $p \in \overline{S_3^\pm} \cap \overline{S_4^\pm}$ .

Thus the material of [23, 24] is sufficient to produce a complete boundary,  $\sigma_\beta$ , and, with some inferences, its classification. The boundary  $\sigma_\beta$  can be considered to be a ‘good’ boundary since none of the global structure is unresolved, i.e. the boundary contains no mixed points.

I now return to the first boundary,  $\sigma \cup \sigma_\alpha$ . By construction each boundary point of  $\alpha$  must cover the surfaces  $S_3^\pm, S_4^\pm, S_5^\pm$  and the intersections of their closures (minus the closures of  $S_6$  and  $S_7$ ). Indeed, Figure 2 of [24] implies that the function  $T$  extends to a bijective function on the boundary of the range of  $\beta$  (see also the last paragraph of page 571 of [24]). Hence, for all  $[(\alpha, U, \{(t, 0, 0, \phi)\})] \in \sigma_\alpha$ , there is a point  $(T, x, Y, \phi)$  lying in any one of the surfaces  $S_3^\pm, S_4^\pm, S_5^\pm$ , or the intersections of their closures (minus the closures of  $S_6$  and  $S_7$ ), such that  $[(\alpha, U, \{(t, 0, 0, \phi)\})] \triangleright [(\beta, U, \{(T, x, Y, \phi)\})]$ . In particular each boundary point in  $\sigma_\alpha$  covers a regular point lying in  $S_5^\pm$ . This justifies the claim that  $\sigma_\alpha$  is composed of directional singularities.

### 3.3. Smooth Gowdy symmetric generalized Taub-NUT spacetimes

The last example comes from Beyer and Hennig's work on the existence, and global properties, of smooth Gowdy symmetric generalized Taub-NUT spacetimes, [5]. A smooth Gowdy symmetric generalized Taub-NUT spacetime is a generalization of a spacetime in Moncrief's class of generalized Taub-NUT solutions of the vacuum Einstein equations on  $(0, \pi) \times \mathbb{S}^3$  with Gowdy,  $U(1) \times U(1)$ , symmetry such that the 'surface' at  $\{0\} \times \mathbb{S}^3$  is a smooth past Cauchy horizon, [5, Section 3.2].

Beyer and Hennig's analysis is based on the study of a set of differential equations. They do not solve these equations to produce a closed form equation for the metric. Nevertheless a great deal can still be deduced about the chart induced boundaries of this class of spacetimes. Due to the lack of a closed form of the metric additional work would be required before other boundary constructions could be applied.

Because of the symmetries of the spacetimes, and because of the choices made with regards to the killing vectors used [5, Section 2.2], the following discussion can be reduced to the coordinates  $t \in (0, \pi)$  and  $\theta \in (0, \pi)$  where  $\theta$  is the coordinate on  $\mathbb{S}^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$  implicitly determined by, [5, Section 2.2],

$$\begin{aligned} x_1 &= \cos \frac{\theta}{2} \cos \lambda_1, & x_2 &= \cos \frac{\theta}{2} \sin \lambda_1, \\ x_3 &= \sin \frac{\theta}{2} \cos \lambda_2, & x_4 &= \sin \frac{\theta}{2} \sin \lambda_2. \end{aligned}$$

Let  $\alpha$  be the chart defined by these coordinates. Let  $U = \mathbb{R} \times (0, \pi)$  so that  $\text{BP}(\alpha) \cap U = \text{BP}(\alpha) = \{0, \pi\} \times (0, \pi)$  and  $(\alpha, U) \in \text{EX}(M)$ .

Beyer and Hennig demonstrate the global existence of solutions to the differential equations by first proving local existence for the Fuchsian system, [5, Equations 35 and 36],

$$\begin{aligned} D^2 S - t^2 \Delta_{\mathbb{S}^2} S &= (1 - t \cot t) DS - \exp(-2S) \left( (D\omega)^2 - (t\partial_\theta \omega)^2 \right), \\ D^2 \omega - 4D\omega - t^2 \Delta_{\mathbb{S}^2} \omega &= (1 - t \cot t) D\omega + 2(DS - 2)D\omega - 2(t\partial_\theta S)(t\partial_\theta \omega), \end{aligned}$$

where  $D = t\partial_t$  and  $\Delta_{\mathbb{S}^2} = \partial_\theta^2 + \cot \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\phi^2$  is the Laplace operator on the unit sphere. The functions  $S, \omega$  are related to the metric components, [5, Section 2 and 3]. Global existence is then implied by [6, Theorem 6.3].

In Section 3 of [5], Beyer and Hennig show that the solution to these equations can be extended to  $(-\epsilon, \pi) \times \mathbb{S}^3$  for some  $\epsilon > 0$ . Note, however, that on  $(-\epsilon, 0) \times \mathbb{S}^3$  the extended solution is no longer guaranteed to be a solution of the vacuum Einstein equations. In any case, the result is that the surface given by  $t = 0$  is composed of  $C^\infty$  regular boundary points. That is the boundary points in  $\{[(\alpha, U, \{(0, \theta)\})] : \theta \in (0, \pi)\}$  are indeterminate points, Definition 2.39. Since the elements  $(0, 0)$  and  $(0, \pi)$  are not boundary points of  $\alpha$  it is not possible to conclude that they are also regular boundary points. This problem has been caused by the coordinate singularity in the definition of  $\alpha$ . Beyer and Hennig's analysis does not suffer this restriction, hence it is clear that if the  $\theta$  axis was rotated to define a new chart the points corresponding to  $(0, 0)$  and  $(0, \pi)$  would be  $C^\infty$  regular.

To study the global behaviour of solution's to the Fuchsian system Beyer and Hennig recast it using an Ernst potential,  $\mathcal{E}$ , that solves the equation, [5, Equation 58],

$$(-\partial_t \mathcal{E} - \cot t \partial_t \mathcal{E} + \partial_\theta^2 \mathcal{E} + \cot \theta \partial_\theta \mathcal{E}) f = -(\partial_t \mathcal{E})^2 + (\partial_\theta \mathcal{E})^2,$$

where  $f$  is the real part of  $\mathcal{E}$ . This is equivalent to a linear partial differential system which reduces to a linear ordinary differential system on each of the surfaces  $t = 0$ ,  $\theta = 0$  and  $\theta = \pi$ , [5, Equation 75]. By solving this linear ODE Beyer and Hennig are able to study the properties of the Ernst potential on the surface  $t = \pi$ . It turns out that the behaviour of the Ernst potential on the surface  $t = \pi$  depends on two parameters  $b_A$  and  $b_B$  which are related to the initial data of the Ernst equation, [5, Section 4.3.2]. Beyer and Hennig make conclusions regarding the  $t = \pi$  surface by dividing the behaviour of the Ernst potential into four cases, [5, Section 4.4];

$b_A = b_B$ : The surface  $t = \pi$  is a  $C^\infty$  regular Cauchy horizon. Thus the boundary points  $(\alpha, U, \{(\pi, \theta)\})$ ,  $\theta \in (0, \pi)$ , are  $C^\infty$  regular boundary points, [5, Section 4.4.1]. As before if a new chart is introduced, by rotating the  $\theta$  axis, the boundary points corresponding to  $(\pi, 0)$  and  $(\pi, \pi)$  would be  $C^\infty$  regular.

$b_A \neq b_B$  **and**  $b_B \neq b_A \pm 4$ : In this case one of the metric components diverges on the surface  $t = \pi$ . The Ernst potential is, however, regular everywhere and the Kretschmann scalar is bounded on  $t = \pi$ . To analyse this further Beyer and Hennig construct a new chart, which will be denoted by  $\beta$ . With respect to this new chart the surface given by  $t = \pi$  is a regular Cauchy horizon, see Equation (117), and the comments immediately before Equation (117), of [5]. Each boundary point  $(\alpha, U, \{(\pi, \theta)\})$  is therefore covered by the set of  $C^\infty$  regular boundary points  $\{(\beta, X, (0, \theta)) : \theta \in (0, \pi)\}$ , where  $(\beta, X)$  is a suitable extension. The boundary points  $[(\alpha, U, \{(\pi, \theta)\})]$  are therefore indeterminate boundary points, Definition 2.39.

Again, Beyer and Hennig's analysis indicates that if the  $\theta$  axis were rotated and the same analysis performed the points corresponding to  $(\pi, 0)$  and  $(\pi, \pi)$  would be removable.

$b_B = b_A + 4$ : In this case the Ernst potential diverges on the surface  $\theta = 0$ , [5, Section 4.4.2]. The Kretschmann scalar on the surface  $\theta = 0$  behaves like  $\frac{1}{(\pi-t)^{12}}$  as  $t \rightarrow \pi$ , [5, Section 4.4.2]. Because of this Beyer and Hennig are unable to use the techniques used in the previous two cases. To avoid this issue they take a sequence of solutions with  $b_B \neq b_A + 4$  that converge to the  $b_B = b_A + 4$  case. The result is that the Ernst potential that is the limit of the Ernst potentials of the sequence of solutions is regular for  $0 < \theta \leq \pi$  and diverges for  $\theta = 0$ , [5, Section 4.4.2]. The implication being that the boundary points  $\{(\alpha, U, \{(\pi, \theta)\}) : \theta \in (0, \pi)\}$  are  $C^\infty$  regular boundary points.

Their analysis also shows that, if the  $\theta$  axis were rotated the point corresponding to  $(\alpha, U, \{(\pi, \pi)\})$  would be a  $C^\infty$  regular point and that the point corresponding to  $(\alpha, U, \{(\pi, 0)\})$  would be an essential singularity. Beyer and Hennig's analysis in [5] does not show if  $[(\alpha, U, \{(\pi, 0)\})]$  is pure or directional.

$b_B = b_A - 4$ : The analysis of this case is exactly the same as for  $b_B = b_A + 4$  except that the divergence occurs in the limit to the point  $(\pi, \pi)$  along the surface  $\theta = \pi$ .

## A. Properties of the completion of a manifold with respect to sets of extensions

In Section 2.3 it was claimed that;

1. the quotient map  $q$  is open, continuous and such that its restriction, for all  $(\alpha, U) \in S_Q$ , to  $N_{(\alpha, U)}$  is a homeomorphism,
2. the space  $Q(M)$  is a  $T_1$ , separable, first countable, locally metrizable topological space, and,
3. that there exists a continuous map  $\iota_Q : M \rightarrow Q(M)$  that is a homeomorphism onto its image so that  $\overline{\iota_Q(M)} = Q(M)$ .

This section presents proofs of these claims. Throughout this section I assume that  $Q \subset \text{EX}(M)$  and that  $N_Q, S_Q, Q(M)$  and  $q : N_Q \rightarrow Q(M)$  are as given in Definition 2.15.

**Proposition A.1.** *The map  $q$  is open.*

*Proof.* It is sufficient to show that  $q^{-1}(q(W))$  is open for any open  $W \subset N_{(\alpha, U)}$ . Suppose otherwise. Then there exists  $(\beta, X) \in S_Q$  so that  $N_{(\beta, X)} \cap q^{-1}(q(W))$  is not open. Since  $N_{(\beta, X)} \cap q^{-1}(q(W))$  is first countable there exists  $x \in q^{-1}(q(W)) \cap N_{(\beta, X)}$  and  $(x_i) \subset N_{(\beta, X)} \setminus q^{-1}(q(W))$  so that  $x_i \rightarrow x$  in  $N_{(\beta, X)}$ . Since  $q(x) \in q(W)$  there exists  $y \in W$  so that  $q(x) = q(y)$ . By the definition of  $Q(M)$  this implies that there exists a subsequence  $(y_i)$  of  $(x_i)$  so that  $\alpha \circ \beta^{-1}(y_i) \rightarrow y$ . Since  $W$  is open there exists some  $i$  so that  $\alpha \circ \beta^{-1}(y_i) \in W$ . By construction there exists some  $x_j$  so that  $x_j = y_i$ . Thus  $q(x_j) = q(\alpha \circ \beta^{-1}(y_i)) \in q(W)$ . This is a contradiction and therefore  $q$  is an open map.  $\square$

**Corollary A.2.** *The space  $Q(M)$  is  $T_1$ .*

*Proof.* Let  $[x], [y] \in Q(M)$ . Then there exists  $(\alpha, U) \in S_Q$  so that  $x \in N_{(\alpha, U)}$ . If  $[y] \in q(N_{(\alpha, U)})$  then there exists  $z \in N_{(\alpha, U)}$  so that  $q(z) = q(y)$ . Since  $N_{(\alpha, U)}$  is Hausdorff and as  $q$  is open it is clear that either  $[x] = [y]$  or  $[x]$  and  $[y]$  are  $T_1$  separated. So suppose that  $[y] \notin q(N_{(\alpha, U)})$ . There exists  $(\beta, X) \in S_Q$  so that  $y \in N_{(\beta, X)}$ . By symmetry we can assume that  $[x] \notin q(N_{(\beta, X)})$ . By construction  $N_{(\alpha, U)}$  and  $N_{(\beta, X)}$  are open subsets of  $N_Q$ . Since  $q$  is open we have the required open sets and  $Q(M)$  is  $T_1$ .  $\square$

**Corollary A.3.** *The space  $Q(M)$  is first countable.*

*Proof.* The space  $N_Q$  is clearly first countable. Since  $Q(M)$  is the image of an open continuous map it is also first countable, [29, Problem 16A.3].  $\square$

**Proposition A.4.** *For all  $(\alpha, U) \in S_Q$  the map  $q|_{N_{(\alpha, U)}}$  is a homeomorphism.*

*Proof.* The map  $q$  is continuous by definition of the topology on  $Q(M)$ . From above  $q$  is open. Surjectivity of  $q|_{N_{(\alpha, U)}}$  follows as the image of this map is  $q(N_{(\alpha, U)})$ . Injectivity follows from the construction of  $Q(M)$  and as each  $N_{(\alpha, U)}$  is Hausdorff.  $\square$

**Corollary A.5.** *The space  $Q(M)$  is locally metrizable.*

*Proof.* Let  $d_{(\alpha, U)} : N_{(\alpha, U)} \times N_{(\alpha, U)} \rightarrow \mathbb{R}$  be the distance on  $N_{(\alpha, U)}$  induced by the euclidean distance on  $\mathbb{R}^n$  and the inclusion  $N_{(\alpha, U)} \subset \mathbb{R}^n$ . Since  $N_{(\alpha, U)}$  was given the relative topology with respect to this inclusion the distance  $d$  is compatible with the topology on  $N_{(\alpha, U)}$ . Define  $d : q(N_{(\alpha, U)}) \times q(N_{(\alpha, U)}) \rightarrow \mathbb{R}$  by

$$d([x], [y]) = d_{(\alpha, U)} \left( \left( q|_{N_{(\alpha, U)}} \right)^{-1}(x), \left( q|_{N_{(\alpha, U)}} \right)^{-1}(y) \right).$$

It can easily be checked that  $d$  is a distance on  $q(N_{(\alpha, U)})$ . Since  $q$  is open  $d$  is compatible with the topology on  $Q(M)$ . Since  $Q(M)$  is covered by the set  $\{q(N_{(\alpha, U)}) : (\alpha, U) \in S_Q\}$ ,  $Q(M)$  is locally metrizable.  $\square$

**Proposition A.6.** *There exists an injective continuous function  $\iota_Q : M \rightarrow Q(M)$  that is a homeomorphism onto its image. The image of  $M$  under  $\iota_Q$  is an open dense subset of  $Q(M)$ .*

*Proof.* For each  $x \in M$  choose some pair  $(\alpha_x, U_x) \in S_Q$  so that  $x \in \text{dom}(\alpha_x)$ , then  $\alpha_x(x) \in N_Q$ . Define  $\iota_Q(x) = q(\alpha_x(x))$ . The definition of  $Q(M)$  implies that  $\iota_Q$  is well defined and independent of the choice of  $\alpha_x$ . In particular, due to the definition of  $Q(M)$ , if  $x \in M$  then  $\iota_Q(x) = [\alpha(x)]$  for any  $\alpha \in \mathcal{A}(M)$  so that  $x \in \text{dom}(\alpha)$ .

Suppose that  $x, y \in M$  are such that  $\iota_Q(x) = \iota_Q(y)$ . That is  $[\alpha_x(x)] = [\alpha_y(y)]$ , by definition this implies that  $\alpha_y \circ \alpha_x^{-1}(\alpha_x(x)) = \alpha_y(y)$ . Thus  $x = y$ . Hence  $\iota_Q$  is injective.

I now show that  $\iota_Q$  is continuous. Let  $V \subset Q(M)$  be open. By definition  $q^{-1}(V)$  is open, hence for each pair  $(\alpha, U) \in S_Q$  the set  $\alpha^{-1}(N_{(\alpha, U)} \cap q^{-1}(V)) \subset M$  is open. I claim that

$$\iota_Q^{-1}(V) = \bigcup_{(\alpha, U) \in S_Q} \alpha^{-1}(N_{(\alpha, U)} \cap q^{-1}(V)).$$

If this is true then  $\iota_Q^{-1}(V)$  is open and therefore  $\iota_Q$  is continuous.

I now prove the claim. Let  $x \in \iota_Q^{-1}(V)$ , then  $x \in M$  hence there exists  $(\alpha, U) \in S_Q$  so that  $x \in \text{dom}(\alpha)$ . Therefore  $\iota_Q(x) = [\alpha(x)] \in V$ . This implies that  $\alpha(x) \in q^{-1}(V)$  and hence  $x = \alpha^{-1}(\alpha(x)) \in \alpha^{-1}(N_{(\alpha, U)} \cap q^{-1}(V))$ . That is  $\iota_Q^{-1}(V) \subset \bigcup_{(\alpha, U) \in S_Q} \alpha^{-1}(N_{(\alpha, U)} \cap q^{-1}(V))$ . Let  $x \in \bigcup_{(\alpha, U) \in S_Q} \alpha^{-1}(N_{(\alpha, U)} \cap q^{-1}(V))$ . Then there is some  $\alpha$  so that  $\alpha(x) \in N_{(\alpha, U)} \cap q^{-1}(V)$  thus  $[\alpha(x)] \in V$ . Since  $\iota_Q$  is independent of the choice of chart,  $\alpha_x$ ,  $\iota_Q(x) = [\alpha(x)]$ . Thus  $x = \iota_Q^{-1}([\alpha(x)]) \in \iota_Q^{-1}(V)$ . Hence the claim holds and  $\iota_Q$  is continuous.

I now show that  $\iota_Q$  is a homeomorphism onto its image. Let  $V \subset M$  be open. In order for  $\iota_Q$  to be a homeomorphism onto its image it is sufficient to show that  $\iota_Q(V)$  is open. By definition this requires that  $q^{-1}(\iota_Q(V))$  is open. I claim that

$$q^{-1}(\iota_Q(V)) = \bigcup_{(\alpha, U) \in S_Q} \alpha(\text{dom}(\alpha) \cap V).$$

If the claim holds, then as  $\alpha(\text{dom}(\alpha) \cap V)$  is open  $q^{-1}(\iota_Q(V))$  must be open. Hence  $\iota_Q$  would be an open map.

I now prove the claim. Let  $x \in q^{-1}(\iota_Q(V))$ . Then  $x \in N_{(\alpha, U)}$  for some  $(\alpha, U) \in S_Q$ . Thus either  $x \in \text{ran}(\alpha)$  or  $x \in \partial_U \text{ran}(\alpha)$ . By assumption  $q(x) = [x] \in \iota_Q(V)$  and therefore  $\iota_Q^{-1}([x]) \in V$ . Since  $V \subset M$  the definition of the equivalence relation defining  $q$  implies that  $x \in \text{ran}(\alpha)$ . Let  $y \in \text{dom}(\alpha)$  be such that  $\alpha(y) = x$ . Then  $q(x) = [\alpha(y)] \in \iota_Q(V)$ . Since  $\iota_Q$  is independent of the choice of  $\alpha_y$  this implies that  $y = \iota_Q^{-1}([\alpha(y)]) \in V$ . Thus  $y \in \text{dom}(\alpha) \cap V$  and so  $x \in \alpha(\text{dom}(\alpha) \cap V)$ . That is,  $q^{-1}(\iota_Q(V)) \subset \bigcup_{(\alpha, U) \in S_Q} \alpha(\text{dom}(\alpha) \cap V)$ . Let  $x \in \alpha(\text{dom}(\alpha) \cap V)$  for some  $(\alpha, U) \in S_Q$ , then  $\alpha^{-1}(x) \in V$ . Since  $\iota_Q$  is independent of the choice of  $\alpha_x$ ,  $\iota_Q(\alpha^{-1}(x)) = [x] \in \iota_Q(V)$ . Thus  $x \in q^{-1}(\iota_Q(V))$  and hence the claim holds.

Since  $\iota_Q$  is continuous it is clear that  $\iota_Q(M)$  is open. It remains to prove that  $\overline{\iota_Q(M)} = Q(M)$ . Let  $[x] \in Q(M)$  so that  $x \notin \iota_Q(M)$ . By definition there exists a pair  $(\alpha, U) \in S_Q$  so that  $x \in N_{(\alpha, U)}$ . If  $x \in \text{ran}(\alpha)$  then  $\iota_Q(\alpha^{-1}(x)) = [x] \in \iota_Q(M)$  which is a contradiction. Therefore  $x \in \partial_U \text{ran}(\alpha)$ . Let  $V \subset Q(M)$  be an open neighbourhood of  $[x]$ . Then  $q^{-1}(V) \cap N_{(\alpha, U)}$  is open in  $N_{(\alpha, U)}$  and hence intersects  $\text{ran}(\alpha)$ . Let  $y \in \text{ran}(\alpha) \cap q^{-1}(V)$ . Then  $\iota_Q(\alpha^{-1}(y)) = q(y) \in V$ . Hence every open neighbourhood of  $x$  in  $Q(M)$  intersects  $\iota_Q(M)$ . Therefore  $\overline{\iota_Q(M)} = Q(M)$  as required.  $\square$

**Corollary A.7.** *The space  $Q(M)$  is separable.*

*Proof.* The manifold  $M$  is separable and  $\overline{\iota_Q(M)} = Q(M)$ . Therefore  $Q(M)$  is separable.  $\square$

## B. Properties of $Q(M)$ with respect to a compatible set of extensions

This section contains the proofs of Proposition 2.29 and Corollary 2.30 which are restated here as Proposition B.1 and Corollary B.2.

**Proposition B.1** (Proposition 2.29). *Let  $\phi : M \rightarrow M_\phi$  be an envelopment. Then, in the notation of Proposition 2.9,  $Q = \{(\alpha \circ \phi, \text{ran}(\alpha)) : \alpha \in \mathcal{A}(M_\phi), \text{dom}(\alpha) \cap \partial\phi(M) \neq \emptyset\}$  is a pairwise compatible set of extensions and there exists a homeomorphism, in the notation of Definition 2.15,  $f : Q(M) \rightarrow \overline{\phi(M)}$  so that  $f \circ \iota_Q = \phi$ ,  $f(q(N_{(\alpha \circ \phi)})) = \text{dom}(\alpha) \cap \overline{\phi(M)}$  and  $\overline{\beta \circ \phi \circ \phi^{-1} \circ \alpha^{-1}} = \beta \circ \alpha^{-1}$ .*

*Proof.* I use the notation of Definition 2.15 throughout this proof.

Let  $(\alpha \circ \phi, \text{ran}(\alpha)), (\beta \circ \phi, \text{ran}(\beta)) \in Q$ ,  $x \in \text{BP}(\alpha) \cap \text{ran}(\alpha)$  and  $y \in \text{BP}(\beta) \cap \text{ran}(\beta)$  such that  $(\alpha, U, \{x\}) \perp (\beta, V, \{y\})$ . Then there exists  $(z_i) \subset M$  so that  $\alpha \circ \phi(z_i) \rightarrow x$  and  $\beta \circ \phi(z_i) \rightarrow y$ . Therefore  $\phi(z_i) \rightarrow \alpha^{-1}(x)$  and  $\phi(z_i) \rightarrow \beta^{-1}(y)$ . Since  $M_\phi$  is Hausdorff  $\alpha^{-1}(x) = \beta^{-1}(y)$  which implies that  $(\alpha \circ \phi, \text{ran}(\alpha), \{x\}) \equiv (\beta \circ \phi, \text{ran}(\beta), \{y\})$ . Thus  $Q$  consists of pairwise compatible extensions.

Define  $\hat{f} : N_Q \rightarrow \overline{\phi(M)}$  by

$$\hat{f}(x) = \begin{cases} \phi \circ \alpha^{-1}(x) & x \in N_{(\alpha, U)}, (\alpha, U) \in P_Q \\ \alpha^{-1}(x) & x \in N_{(\alpha \circ \phi, \text{ran}(\phi))}, (\alpha \circ \phi, \text{ran}(\alpha)) \in Q. \end{cases}$$

It is clear that  $\hat{f}$  is well defined.

I now show that  $\hat{f}$  descends to  $Q(M)$ . Let  $x, y \in N_Q$  so that  $[x] = [y]$ , I need to show that  $\hat{f}(x) = \hat{f}(y)$ . We have three cases to check. **Case 1.** Let  $x \in N_{(\alpha, U)}$  with  $(\alpha, U) \in P_Q$  and  $y \in N_{(\beta, V)}$  with  $(\beta, V) \in P_Q$ . Since  $[x] = [y]$ , by construction,  $\beta \circ \alpha^{-1}(x) = y$ . Thus  $\hat{f}(x) = \hat{f}(y)$ . **Case 2.** Let  $x \in N_{(\alpha, U)}$  with  $(\alpha, U) \in P_Q$  and  $y \in N_{(\beta \circ \phi, \text{ran}(\beta))}$  with  $(\beta \circ \phi, \text{ran}(\beta)) \in Q$ . Since  $[x] = [y]$ , by construction,  $\beta \circ \phi \circ \alpha^{-1}(x) = y$ . Thus  $\hat{f}(x) = \phi \circ \alpha^{-1}(x) = \beta^{-1}(y) = \hat{f}(y)$ . **Case 3.** Let  $x \in N_{(\alpha \circ \phi, \text{ran}(\alpha))}$  with  $(\alpha \circ \phi, \text{ran}(\alpha)) \in Q$  and  $y \in N_{(\beta \circ \phi, \text{ran}(\beta))}$  with  $(\beta \circ \phi, \text{ran}(\beta)) \in Q$ . If  $x \in \text{dom}(\alpha \circ \phi)$  then as  $[x] = [y]$ , by construction,  $\beta \circ \phi \circ \phi^{-1} \circ \alpha^{-1}(x) = y$  and thus  $\hat{f}(x) = \alpha^{-1}(x) = \beta^{-1}(y) = \hat{f}(y)$ . If  $x \notin \text{dom}(\alpha \circ \phi)$  then  $x \in \partial_{\text{ran}(\alpha)} \text{ran}(\alpha \circ \phi)$ . Choose  $(x_i) \subset \text{ran}(\alpha \circ \phi)$  so that  $x_i \rightarrow x$ . Since  $[x] = [y]$  there exists a subsequence  $(y_i) \subset (\phi^{-1} \circ \alpha^{-1}(x_i))$  so that  $(y_i) \subset \text{dom}(\beta \circ \phi)$  and  $\beta \circ \phi(y_i) \rightarrow y$ . Thus there exists a sequence  $(\alpha^{-1}(x_i))$  of  $M_\phi$  so that  $\alpha^{-1}(x_i) \rightarrow \alpha^{-1}(x)$  and so that  $\beta^{-1}(y)$  is a limit point of  $(\alpha^{-1}(x_i))$ . Since  $M_\phi$  is Hausdorff this implies that  $\alpha^{-1}(x) = \beta^{-1}(y)$ . Thus  $\hat{f}(x) = \hat{f}(y)$ . Hence  $\hat{f}$  descends to a function  $f : Q(M) \rightarrow \overline{\phi(M)}$ , so that  $f \circ q = \hat{f}$ .

It is clear that  $\hat{f}$  is continuous, thus  $f$  is continuous. To check that  $f$  is open it suffices to prove that  $f(q(V))$  is open for any  $V \subset N_{(\alpha, U)}$  an open subset with  $(\alpha, U) \in S_Q$ . But  $f \circ q|_{N_{(\alpha, U)}} = \hat{f}|_{N_{(\alpha, U)}}$  which is either  $\phi \circ \alpha^{-1}$  or  $\alpha^{-1}$  both of which are open maps. Therefore  $f(q(V))$  is open in the subspace topology of  $\hat{f}(N_{(\alpha, U)})$ . Since  $\hat{f}(N_{(\alpha, U)})$  is open in  $\overline{\phi(M)}$  the set  $f(q(V))$  is open in  $\overline{\phi(M)}$ . Hence  $f$  is open. Suppose that  $f([x]) = f([y])$ , where  $x \in N_{(\alpha, U)}$  and  $y \in N_{(\beta, V)}$ . Then there exists  $(z_i) \subset \phi(M)$  so that  $z_i \rightarrow f([x])$  and  $z_i \rightarrow f([y])$ . By construction of  $f$ ,  $f \circ q = \hat{f}$ . Thus as both  $f$  and  $q$  are open  $\hat{f}$  is open. Since  $\hat{f}$  is onto there exists  $(x_i) \subset \text{dom}(\alpha)$  and  $(y_i) \subset \text{dom}(\beta)$  so that  $x_i \rightarrow x$ ,  $y_i \rightarrow y$ ,  $\hat{f}(x_i) = \hat{f}(y_i)$  and  $(\hat{f}(x_i)) \subset (z_i)$ . This is sufficient to show that  $(\alpha, U, \{x\}) \perp (\beta, V, \{y\})$ . By assumption on  $Q$  this implies that  $[x] = [y]$  so that  $f$  is injective. Let  $x \in \overline{\phi(M)}$ . If there exists  $y \in M$  so that  $\phi(y) = x$  then there exists  $\alpha \in \mathcal{A}(M)$  so that  $\alpha(y) \in N_{(\alpha, \text{ran}(\alpha))} \subset N_Q$ . Thus  $f(y) = \phi \circ \alpha^{-1}(\alpha(y)) = x$ . Otherwise there exists  $\alpha \in \mathcal{A}(M_\phi)$  so that  $x \in \text{dom}(\alpha)$ . Thus  $(\alpha \circ \phi, \text{ran}(\alpha)) \in Q$  and  $\alpha(x) \in N_Q$ . By construction  $f([\alpha(x)]) = \alpha^{-1} \circ \alpha(x) = x$ . Hence  $f$  is bijective.

I now show that  $f \circ \iota_Q = \phi$ . Let  $x \in M$  then there exists  $\alpha \in \mathcal{A}(M)$  so that  $x \in \text{dom}(\alpha)$ . Hence  $f \circ \iota_Q(x) = f(q(\alpha(x))) = \hat{f}(\alpha(x)) = \phi \circ \alpha^{-1}(\alpha(x)) = \phi(x)$ .



I now show that  $f(q(N_{(\alpha \circ \phi, \text{ran}(\alpha))})) = \text{dom}(\alpha) \cap \overline{\phi(M)}$  for  $(\alpha \circ \phi, \text{ran}(\alpha)) \in Q$ . By definition  $f(q(N_{(\alpha \circ \phi, \text{ran}(\alpha))})) = \alpha^{-1}(N_{(\alpha \circ \phi, \text{ran}(\alpha))})$ . But, by definition,  $N_{(\alpha \circ \phi, \text{ran}(\alpha))} = \alpha(\text{dom}(\alpha) \cap \phi(M)) \cup \partial_{\text{ran}(\alpha)} \alpha(\text{dom}(\alpha) \cap \phi(M))$ . Since  $\text{ran}(\alpha)$  and  $\alpha(\text{dom}(\alpha) \cap \phi(M))$  are open  $\partial_{\text{ran}(\alpha)} \alpha(\text{dom}(\alpha) \cap \phi(M)) = \text{ran}(\alpha) \cap \partial \alpha(\text{dom}(\alpha) \cap \phi(M))$ . Thus to give the set equivalence it is necessary and sufficient to show that  $\alpha(\text{dom}(\alpha) \cap \partial \phi(M)) = \text{ran}(\alpha) \cap \partial \alpha(\text{dom}(\alpha) \cap \phi(M))$ . This set equality follows directly from the definitions.

Lastly, I show that  $\overline{\beta \circ \phi \circ \phi^{-1} \circ \alpha} = \beta \circ \alpha^{-1}$ . Suppose that  $\overline{\beta \circ \phi \circ \phi^{-1} \circ \alpha}(x) = y$ , where  $x \in N_{(\alpha \circ \phi, \text{ran}(\alpha))}$  and  $y \in N_{(\beta \circ \phi, \text{ran}(\beta))}$ . By definition  $q(x) = q(y)$ . Thus  $\alpha^{-1}(x) = \hat{f}(x) = f(q(x)) = f(q(y)) = \hat{f}(y) = \beta^{-1}(y)$  and so  $\beta \circ \alpha^{-1}(x) = y$ . The reverse implication,  $\beta \circ \alpha^{-1}(x) = y$  implies  $q(x) = q(y)$ , holds by definition of  $Q(M)$ . Hence  $\overline{\beta \circ \phi \circ \phi^{-1} \circ \alpha^{-1}(x)} = y$ , as required.  $\square$

**Corollary B.2** (Corollary 2.30). *Let  $\phi : M \rightarrow M_\phi$  be an envelopment and let  $Q(M)$  be as given in Definition 2.15. If  $\overline{\phi(M)}$  is a manifold with boundary then  $Q(M)$  is a manifold with boundary and  $f$  is a diffeomorphism, where  $f$  is given in Proposition 2.29.*

*Proof of Corollary 2.30.* I use the notation of Proposition 2.29. Since  $f$  is a homeomorphism  $Q(M)$  is second countable and Hausdorff as  $\overline{\phi(M)}$  is second countable and Hausdorff. Let  $\mathcal{A}(Q(M)) = \{\beta \circ f : \beta \in \mathcal{A}(\overline{\phi(M)})\}$ . The transition maps between charts are  $\beta \circ f \circ f^{-1} \circ \alpha^{-1} = \beta \circ \alpha^{-1}$  and therefore  $Q(M)$  is a manifold with boundary. Note that this atlas is essentially that given by the extensions in  $Q$  due to the definition of  $f$  in Proposition 2.29. In order for  $f$  to be a diffeomorphism it must be the case that all  $\alpha \in \mathcal{A}(Q(M))$  and all  $\beta \in \mathcal{A}(\overline{\phi(M)})$ ,  $\beta \circ f \circ \alpha^{-1}$  must be a diffeomorphism. By definition there exists  $\gamma \in \mathcal{A}(\overline{\phi(M)})$  so that  $\alpha = \gamma \circ f$ . Thus  $\beta \circ f \circ \alpha^{-1} = \beta \circ f \circ f^{-1} \circ \gamma = \beta \circ \gamma$  which is a diffeomorphism by assumption on  $\overline{\phi(M)}$ .  $\square$

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